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THE
DOCTRINE
OF
ANNUITIES and ASSURANCES
ON
LIVES and SURVIVORSHIPS,
STATED AND EXPLAINED.

THE
DOCTRINE



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THE
DOCTRINE
OF
ANNUITIES and ASSURANCES
ON
LIVES and SURVIVORSHIPS,
STATED AND EXPLAINED.

By WILLIAM MORGAN,
ACTUARY to the Society for EQUITABLE ASSURANCES
on LIVES and SURVIVORSHIPS.

To which is added,
AN INTRODUCTION,
Addressed to the SOCIETY.

ALSO
An ESSAY on the PRESENT STATE of
POPULATION in ENGLAND and WALES.

By the Reverend Dr. PRICE.

LONDON:
Printed for T. CADELL, in the Strand.
M.DCC.LXXIX.

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ANNUITIES AND ASSURANCES

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LONDON

Printed by T. Cadell, in the Strand



INTRODUCTION.

CONTAINING

Observations addressed to the Society
for Equitable Assurances on Lives
and Survivorships.

HAVING encouraged Mr. *Morgan* to undertake this work, and having also, by his desire, revised it with some care, I think myself obliged to introduce it to the notice of the public, by giving the following account of its contents, and of the views with which it has been composed.

The *first* chapter contains an explanation of the nature of Assurances on Lives and Survivorships; together with a particular account of the Society for making such Assurances, and of the methods which have been taken to determine how far it has been hitherto a gainer by the business it has transacted.

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The *second* chapter contains an explanation of the Doctrine of Life-Annuities in general, and of the principles on which their values are calculated. At the end of this chapter, an account is given of a method of expediting all calculations of the values of Life-Annuities, which must, I think, be very acceptable to all who have ever employed themselves in making such calculations.

In these two chapters Mr. *Morgan* has studied to render the subjects of which he treats as intelligible as possible to persons who may be unacquainted with mathematics.

The *third* chapter contains a complete account of the Rules for solving all questions concerning the values (in single and annual payments) of all Annuities, whether in possession or expectation; and of all Reversionary Interests depending on any *one*, *two*, or *three* lives, or on any Survivorships among them, either for *Terms* or for *Perpetuity*. Several of these questions are incapable of being answered without entering into the abstrusest parts of mathematics. Most of them have never been before answered; and yet there arises in many cases a necessity of answering them, in order to

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determine correctly the values of *Reversionary Interests* depending on *Survivorships*; and some of them are often occurring in practice at the office whose business *Mr. Morgan* transacts.—All the Solutions are given in words at length, with suitable examples; but the mathematical investigations have been thrown together into the Appendix; where every rule or direction, about which any doubt can be entertained, is particularly demonstrated.—Every competent judge who will examine this chapter, and compare it with the demonstrations in the Appendix, must admire *Mr. Morgan's* attention and skill; and see that it contains a valuable addition to this part of science.

At the close of the whole, I have, with *Mr. Morgan's* leave, inserted an essay, containing an account of the present State of Population in this kingdom; and the intention of which is, to state a great variety of facts which seem to prove, that while neighbouring kingdoms are increasing, this kingdom has for many years been decreasing fast, in consequence of the operation of some of the worst causes of depopulation; and, particularly, our increased debts and luxury.

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Such are the contents of this work ; and it is now offered to the public chiefly for the following reasons.

First. In the course of the traffic of this kingdom, and particularly the law transactions of it, such questions as those contained in this work are continually occurring; and, in numberless cases, it is impossible to give right decisions, or to make an equitable distribution of property, according to the real value of different claims and interests, without obtaining accurate answers to them. The experience of many years has taught me that there are great mistakes committed in answering these questions, by which many persons are injured in their property. It seems, therefore, that it may be of particular use to the public to be informed where to apply for solutions that may be depended upon ; and, at the same time, to be furnished with such an account of the rules and principles of calculation, mathematically demonstrated, as may remove every doubt, and be a sufficient direction to all who may chuse to employ themselves in business of this kind.

But perhaps this reason alone might not have led Mr. *Morgan* to undertake the labour of this work. There is another reason

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son by which he has been more immediately and strongly influenced.—He is employed in transacting the business of a Society which *assures* all kinds of Reversionary Annuities, and contingent interests, dependent on the continuance of any Lives or any Survivorships of any Lives beyond other Lives. In carrying on this business the Society professes to regulate its demands by calculations founded on strict mathematical principles. There is not any other society now existing of the same kind; and its business has been, for some years, increasing so fast, that it is already become an object of vast importance to the public, and is likely soon to become of much more importance. It seems, therefore, necessary that the public should be fully informed of its state, the security it offers, the principles on which it proceeds, and the method in which it makes its calculations: And the great design of this publication is to give this information—an information which is rendered the more necessary by the multitude of *bubble*-societies which were some time ago established, and some of which are still existing in this kingdom.

My acquaintance with this Society was occasioned by the publication (in 1770) of
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the first edition of my Treatise on Reversionary Payments. In that Treatise I stated particularly the plan of the Society ; and at the same time offered some observations upon it. The Directors were pleased to attend to these observations; and they have ever since desired my assistance, whenever they have thought I could be of use to them. In return for their regard and attention, as well as from a desire to promote the public interest, I have been anxious about doing them all the service, and giving them all the information, in my power. This led me four years ago to take into consideration the method in which they kept their accounts, and determined annually their profits. Thinking this method too loose, I took the liberty to propose another which I thought more strict and decisive. This method was approved and adopted ; and by means of it the Society has been enabled to keep constantly under its eye the true state of its affairs, the expences of management which it can afford, the progress of the balance in its favour from year to year, and the clear amount remaining in hand of all the profits it has made from the time of its first establishment.—In this method a particular enquiry was (in 1776) made into the state of
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of the Society; and the result was, as explained by Mr. *Morgan* in the second chapter, that it appeared, that a much smaller proportion of the persons assured had died than the calculations supposed; and that the Society possessed *then* a surplus of *income* (or an income more than was necessary to enable it to make good its engagements and to bear the expences of management) equal to 2,400 *l. per annum*, nearly, and a surplus of *stock* equal to 30,000 *l.** In these circumstances, the Society, not willing to continue any exorbitant profits, determined to reduce all the payments for Assurances on Single Lives ONE TENTH. They might have reduced them much more, but there were several reasons which deterred the Directors from proposing any greater reduction.

First. It was proper to give the plan of the Society a longer trial before any great reductions were made in its income. It could not be depended on that the probabilities of living in the society would con-

* The Society has *continued* to increase so fast, that both its income and stock have been in the last three years (or from 1776 to 1779) nearly doubled. Its profits also have in the same time increased considerably, but not in the same proportion, on account of the reductions mentioned in this and the next pages.

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tinue so high as they had hitherto been; and it was to be expected, that seasons of extraordinary mortality would come which would bring on the Society extraordinary expences. Its security against this danger lies entirely in its *surplus stock*; and it was, therefore, necessary, in order to give it full security, and to render it as much as possible a permanent benefit to the public, that this *surplus stock* should be allowed to go on increasing for some years longer. When I speak of *full security*, I must be understood to mean all the security that property in the public funds can give. It is earnestly to be wished this was greater than it is; but though greater might be obtained, yet in an undertaking of this kind it is scarcely reasonable to aim at it. The failure of the public funds will be the commencement of a new æra in this kingdom, of which, like the end of the world, we can now form no conception; and were every one to act with a view to its being so near as perhaps it is, there would be an end of most of the business and traffic of the nation.

Secondly. When the reduction I have mentioned was resolved on, it was also agreed, that the whole overplus should be returned of the payments which had been made

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made by the members, above those which they *would* have made had the reduction taken place at the time of their admission. This made the Society less capable of bearing *any* reduction of its income; and it would have made a *greater* reduction particularly imprudent.—Different opinions have been entertained of this measure; but the truth is, that (however safe and just the prosperous state of the Society then rendered it) it is in itself a measure of the most pernicious tendency, and which ought by no means to be again adopted. A repetition of it in future reductions might hurt the Society essentially, by withdrawing from it that security which it has been providing for many years, and bringing it back to infancy and weakness.—It would likewise be subversive of its own end by making the Society incapable of proceeding to farther reductions as soon as it otherwise might; and by rendering them, when made, less considerable than they should otherwise have been. Such a measure, therefore, might be no less detrimental to the interest of even the members who might be led to desire it by the hope of present profit, than unkind and injurious to the Society at large.—For these reasons, and
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in order to prevent the future bad effects which may arise from the late precedent, I cannot help wishing that a resolution of a General Court could be procured, prohibiting every step of the same kind hereafter.—Could such a resolution be obtained, the Directors would be at liberty to propose such farther reductions as from time to time they might judge proper; and, particularly, they might immediately take off from all the payments for Assurances on Single Lives the addition of 6 *per cent.* with which these payments are at present charged in the Tables. This was once a proper charge; but, in the present circumstances of the Society, it is certainly an exorbitant charge; nor does there appear to be any good reason for * continuing it, except the danger of being forced to another return of over-payments.

Believing as I do, that this Society is capable of being made an object of great consequence not only to *this*, but to *other* kingdoms, and wishing to contribute all I can

* I must be understood to mention this only on the supposition that the Society will chuse to go on using its present Tables.

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to its credit and honour, I hope I shall not be thought too presuming if I proceed a little farther in these strictures, and take the liberty to recommend to it the following observations and proposals.

First. It is, particularly, to be wished that the Society was furnished with a set of Tables (shewing the values of Assurances on Single Lives) more correct and elegant than those it now uses, and more accommodated to the business it carries on. Its present Tables have been calculated from the rate of mortality among the inhabitants of *London*, taken in the gross, during twenty-three years, from 1728 to 1750—a period of time which included two years* of greater mortality than has been ever known in *London* since the plague in 1665. In consequence of this, the values of Assurances on Lives are given in these Tables *somewhat* too high for the inhabitants at large of even *London* itself; and *much* too high for the better part of the inhabitants. There were good reasons for beginning with such Tables; but the Society being now established in some degree of security, and having better grounds to go upon, it would, in my opinion, be

* The years 1740 and 1741.

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right to calculate new Tables founded on observations which will give the values of Life-Affurances, not among the bulk of people in *London*, where life is particularly short, but among mankind in general.—The decrements of life at every age, as deduced by Dr. *Halley* from observations at *Breslaw* in *Silefia*, or those deduced from the bills of mortality at *Northampton* or *Norwich*, are nearly the mean decrements between those in great towns and in country parishes and villages. As the Society assures town and country lives indiscriminately, these are the observations by which it should be guided. But observations properer for its purpose than even these may be now obtained. I mean those furnished by the *Register of Mortality* established a few years ago at *CHESTER*, under the direction of the ingenious Dr. *HAYGARTH*.—*Chester* is an old and a very healthy town, of moderate size, which has continued much the same as to populousness for a long course of years. These are circumstances which render it a situation particularly fitted for shewing the true law that governs the waste of human life in all its stages. The Register which has been established there is more minute and correct than any other ; and it is the only
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one I am acquainted with that gives the difference between the chances of living among males and females, and from which it is possible to compute, with any degree of precision, the values of lives *before* five and *after* seventy years of age. Tables, therefore, of the values of Life-Annuities, Assurances and Reversions calculated from this Register, would be a valuable acquisition; and might form a standard for the business of this Society, which would be of great use to it. The prices of Assurances in such Tables would indeed be found considerably lower than those in the Tables it now uses*. But they might be raised as high as should appear to be necessary to the perfect safety of the Society, by making a charge upon them of 10 or 15 *per cent.* and giving them, in the Proposals of the Society, with such a charge.

* Care should be taken to remember that I have here in view only Assurances on Single Lives, the longest of Two or more Lives, and Joint Lives. The values of Assurances on Survivorships are, in many cases, *greater* when the chances of life are *greater*; and it is of particular consequence that the Society should attend to this, and distinguish between the cases in which its interest requires it to compute from observations that give the chances of life *high*, rather than from those which give them *low*.

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Secondly.

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Secondly. The business of the Society requires that it should be possess of Tables of the values of *two joint* lives at all ages, agreeably to the best observations, and true to at least three places of decimals. Without such Tables it is impossible to find, in many cases, the true values of the Assurances to be made by the Society, and particularly of Assurances on *Survivorships for Terms*.—There are now no such Tables extant. Mr. *Simpson's* Table in the *Select Exercises*, page 266, is adapted only to *London*; and gives the values only to one place of decimals. The Table in the *Treatise on Reversionary Payments*, page 328, is calculated from Mr. *De Moivre's* Hypothesis; and this Hypothesis, though agreeing nearly in the middle stages of life with the *Breslaw* and *Northampton* observations, yet differs so widely from all real observations *before* twenty and *after* seventy, as to be totally improper for use. The Society, therefore, in order to a more perfect management of its business, should direct such Tables as I have described to be calculated. And were they even to go so far as to furnish themselves with Tables, equally exact,

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of the values of *three**, as well as of *two joint* Lives, their business would be still more assisted, and considerable service done to this part of science. The expence of such calculations can be no object to them; and the most easy and expeditious method of making them has been distinctly described by Mr. *Morgan* in the following work.

Thirdly. I must take the liberty to remind the Society of the particular care necessary to prevent the intrusion of bad lives. It must be ruined, should it ever become, not a resource for the living and healthy, but a refuge for the sick and dying. Its prosperous state proves that the vigilance of the Directors has hitherto guarded it against this danger; but its great increase exposes it to more and more of this danger; and therefore it would not perhaps be amiss to appoint a medical assistant, whose particular business it shall be to enquire into the state of health of the persons who are offered to be assured. Should such a regulation be the means of excluding but two

* In making many calculations for the business of the Society, it is necessary to find the values of the *longest* of *two* or of *three* given lives. But this cannot be done without knowing the values of the *two* or of the *three joint* lives.

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or three bad lives annually, which would otherwise have been admitted, the expence attending it will be more than compensated.

This, without doubt, is the *principal* danger to which the Society is exposed. But there is another of considerable consequence about which I cannot be silent; I mean the danger of employing unskilful calculators. Such Tables as I have proposed would lessen this danger; but no Tables can entirely remove it, because there are many cases in which the values of Assurances cannot be obtained by Tables, without nice, and sometimes long and laborious computations, which none but able mathematicians can make. There are indeed (as I have before intimated) no questions, the solution of which requires a stricter attention or more skill in investigation than some in the doctrine of Assurances. Difficult questions of this kind are sometimes brought to the Society; and the Directors, not being themselves mathematicians, are under a necessity, in making their demands, of being governed by their Actuary: And should he happen to be unqualified, he must make mistakes, and either the Public or the Society will be injured.

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In short. The Society can scarcely be sensible enough of the importance of both ability and probity in the servants it employs; nor, therefore, of the particular reason there is for guarding their places against the applications of candidates who, on any future vacancies, may endeavour to intrude themselves by their connexions or influence.

I intended to have proceeded to some farther observations and proposals; but I have perhaps already gone too far. Those which have been offered are sincerely meant to promote the prosperity of the Society; and they will therefore, I doubt not, be received with candour.

RICHARD PRICE.

NEWINGTON-GREEN,

May 29, 1779.

ERRATA.

Page 3, line 11, for "*those of*" read "*in*"

Page 155, line 6th of the Note, for "17th Problem,"
read "19th Problem."

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CHARACTERS

CHARACTERS *used in this Work.*

+ Signifies that the quantity to which it is prefixed is to be added. Thus $\frac{35}{36} + \frac{25}{26}$ denotes the *sum* arising from the addition of $\frac{25}{26}$ to $\frac{35}{36}$.

— Serves to shew that the quantity which it precedes is to be subtracted. Thus $\frac{35}{36} - \frac{25}{26}$ denotes the *difference* between those two fractions, or the *excess* of $\frac{35}{36}$ above $\frac{25}{26}$.

× Signifies that the quantities between which it is placed are to be multiplied together. Thus, $\frac{35}{36} \times \frac{25}{26}$ expresses the product arising from the multiplication of $\frac{35}{36}$ into $\frac{25}{26}$.

= Denotes equality. Thus, $3 + 2 = 5$ signifies that 2 added to 3 is equal to 5.

When a line is drawn over any number of figures, it signifies that all those figures are to be considered as one compound quantity. Thus, $1 - \frac{35}{36} \times 1 - \frac{25}{26}$ denotes the product arising from the multiplication of $\frac{35}{36}$ subtracted from unity, into $\frac{25}{26}$ subtracted from unity.



CHAPTER I.

An Introduction to the Doctrine of Assurances on Lives.

SECT. I.

THE ease with which the value of an Assurance is computed depends on the number of lives involved in the calculation. In order, therefore, to give a clear idea of this subject, and the principles from which the several questions are investigated, it will be necessary to begin with those cases where only one life is concerned.—The simplest of this kind is the Assurance of a single life for the term of one year.—Nothing further is required here than to determine the probability of the life's failing in this time; and in proportion as this probability is greater or less, the value of the

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Affurance will also be greater or less; so that was it *certain* that the life would fail in the year, the premium of Affurance would be equal to the sum assured: was it an *even* chance whether it failed or not, it would be worth *half* as much: was there only *one chance out of twenty*, the premium would only be a twentieth, and so on.

Suppose, for example, that 50 persons aged 39 agreed to assure 100 *l.* each on their lives for one year.—By Dr. *Halley's* Table of Observations it appears that one of them will die in the year. The value of the Affurance, therefore, will be $\frac{1}{50}$ of 100 *l.* or 2 *l.* from each person; for this contribution will just furnish money enough to pay the claim. But since the money is generally advanced at the beginning, and the claim is not paid till the end of the year, this sum must be discounted for a year, that the several contributions, together with their interest, may be just equal to the sum assured.—Had these persons been 59 years of age, 2 of them would have died in the year, and the contributions would have been twice as much; had they been 70 years of age, 4 of them would have died, and consequently the contributions would have been 4 times

as much. This shews the reason why the value of an Assurance for a year is continually increasing with age, till at last it becomes equal to the sum assured.

Let the time now be 2, 3, or any certain number of years, the sum, as before, 100*l*. the whole price of Assurance to be paid down immediately, and the Claim whenever the death happens. — The principles of investigation in this case are a little more complicated than those of the former, and they will perhaps be best explained in the following manner :—Let 86 be supposed the extremity of life, and the difference between this period and the age of the person be called the *complement* of life; let the probability of living also decrease according to *De Moivre's* Hypothesis * in arithmetic pro-

* In this Hypothesis the limit or utmost extent of life is confined to the age of 86, and the probabilities of existence are supposed to decrease in *arithmetic* progression; that is, supposing any number living at a given age, an equal number of them will die every year, till at the age of 86 all the lives will become extinct. Thus for instance; if there are 56 persons living at the age of 30, one of them, by this hypothesis, will die annually during the term of 56 years, at which time the last surviving life will have failed :—Or in other words, 55 will be living at the end of the first year, 54 at the end of the second year, 53 at the end of the third year, and

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progreſſion. Theſe things being premiſed, ſuppoſe the age of the perſon to be aſſured is 66, the complement of his life will be 20; and as the probability of his living is leſſened every year by unity, the chances for his living the firſt year will be 19 out of 20; for his living 2 years, 18 out of 20; for his living 3 years, 17 out of 20, and ſo on; from whence it follows, that the chance for his *dying* the firſt year will be 1 out of 20; for his dying the ſecond year, after ſurviving the firſt, 1 out of 19; of his dying the third year, after ſurviving the two firſt years, 1 out of 18, and ſo on. The value, therefore, of the Assurance the firſt year will be a *twentieth* of the ſum diſcounted for a year. Should the perſon ſurvive this period, for which he has 19 chances out of 20 (or a probability expreſſed by the fraction $\frac{19}{20}$) the value, at the beginning of the

ſo on for 56 years; from whence it follows, that the probability of one perſon's living to the end of the firſt, or ſecond, or third year will be $\frac{55}{56}$, $\frac{54}{56}$, $\frac{53}{56}$ reſpectively; where it appears that the fractions, and conſequently the probabilities they expreſs, decrease *arithmetically*. Therefore if n be the difference between any given age and 86, the probability of exiſting at this age, for one, two, three, &c. years, will be properly denoted by the correſponding term of the ſeries, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c.

first year, of the Assurance in the second year, will depend upon the contingency of his dying in that year after having survived the first. The probability that he dies in the second year, is expressed by the fraction $\frac{1}{19}$, which being multiplied by $\frac{19}{20}$ (the probability that he survives the first) we have $\frac{1}{20}$ for the ratio in which the sum discounted for two years is to be diminished in order to obtain the value of the Assurance for the second year. In the same manner, the value at the beginning of the first year of the Assurance in the third year depends upon the person's surviving the two first years, for which he has 18 chances out of 20, and his dying in the third year, the chance for which is 1 out of 18, these two fractions, or $\frac{18}{20}$ and $\frac{1}{18}$ being multiplied into each other, we still shall have $\frac{1}{20}$ to express the ratio in which the sum, discounted in this case for three years, is to be diminished. By reasoning in the same way, the value of the Assurance during the fourth, fifth, &c. years will be constantly found $\frac{1}{20}$ th of the sum discounted for four, five, &c. years; and the amount of all these sums will be the whole value of the Assurance in one

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present payment.—Was this Assurance extended to the whole duration of life, the value would be determined in a similar manner, only continuing the calculations for a number of years equal to the *complement* of the life; at which time it becomes a certainty by the hypothesis * that it will fail, and of course that the claim will become due.—The same reasoning may be applied to every age, and any table of observations.—Thus; suppose the person is 30, and the probabilities of life as given by Dr. *Halley*. The chances for his living one year will be 523 out of 531; for his living two years, 515 out of 531; for his living three years, 507 out of 531, and so on. The chances for his *dying* the first year are 8 out of 531; for his dying the second year, after surviving the first, 8 out of 523; for his dying the third year, after surviving the two first years, 8 out of 507, &c. The value of the Assurance, therefore, the first year will be 100 *l.* discounted for a year, and diminished in the ratio of $\frac{8}{531}$; the value of it the second year, at the beginning of the first, will be 100 *l.* discounted for two years, and diminished in the ratio of $\frac{523}{531} \times \frac{8}{523}$ or $\frac{8}{531}$; the third year it will be 100 *l.* discounted

* See Note, page 3.

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for three years, and diminished in the ratio of $\frac{507}{531}$ multiplied into $\frac{8}{507}$, or $\frac{8}{531}$; which will continue invariably the same as long as the probabilities of life decrease in arithmetic progression; but this not being always the case, the ratio will consequently be different; hence in the present instance the probability of the person's living 14 years is expressed by the fraction $\frac{407}{531}$ *, and the probability of his dying the 15th, after surviving the first fourteen years, by $\frac{10}{407}$; these two fractions, therefore, being multiplied into each other, we have $\frac{10}{531}$ for the ratio in which the sum (properly discounted) is to be diminished the 15th year. In other years this ratio will be found as much less as it is here greater than $\frac{8}{531}$; so that upon the whole it coincides very nearly with Mr. *De Moivre's* hypothesis. In what manner the amount of the several sums (discounted and diminished for the respective years) is determined, may be seen in Dr. *Price's* Treatise, Question 10th and 14th †; and in *Simson's* Select Exercises, Problem 26. My design being only to give

* See Note under Prob. 4th, chap. 3d.

† These two Questions are the same with the eighth and ninth Problems in the third chapter of this work.

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a general idea of the *principles* from which the values of Assurances are investigated, it is unnecessary to be more particular.

If, instead of a *single* payment, the value of an Assurance of 100 *l.* for any term of years was required in *annual* payments, the answer may be obtained by reasoning in the following manner :—Suppose 50 persons aged 36 had agreed to assure 100 *l.* each on their lives from year to year for any certain term, that is, by equal contributions to discharge whatever claims may happen in the course of every year. By *De Moivre's* hypothesis one of them will die in the first year and become a claimant at the end of the year. The value of the Assurance from each person, therefore, will be 2 *l.*; but if the money has been advanced at the beginning of the year this value will be less, or only 2 *l.* discounted for a year.—The second year there will be 49 persons alive, and one in 49 will die by the same hypothesis, consequently one will become again a claimant of 100 *l.* in the course of the second year; but a contribution of 2 *l.* each from 49 persons will not be sufficient to discharge such a claim; at the beginning of the second year, therefore, they ought to contribute a little more than they did at the beginning
of

of the first year;—and for the same reason, those of them that will live to the *third* year ought to contribute more than they did the *second* year in order to make good the same Assurance, and so on for every year during the whole term. If the price of Assurance is paid, not in annual contributions continually *increasing* from year to year, but in one *fixed* annual contribution, it is obvious that this contribution ought to be higher than the contribution due for an Assurance in the first year of the term, and lower than the same contribution in the last year. What this *mean* should be, may be determined by the Rule in Dr. Price's 14th Question.

Had there been 50 persons at the age of 61, two of them would have died by the same hypothesis every year, and consequently the contributions must have been *double* what they were at the age of 36. From whence the reason may be inferred, why the difference between the price of Assurance for one year, and the short term of six or seven years, is not so considerable on a *young* life as it is on an *old* one; for the probabilities of dying being less in one case than the other, the prices of Assurance must

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must be less, and consequently the *differences* less.

Were these annual payments to be continued during the *whole* of life, that is, was it an Assurance of 100 *l.* to be paid whenever the life fails, the reasoning will be exactly similar to that in the preceding case. But the *mean* of all the payments will be much more considerable here; for those contributions must increase annually, till at last they become equal to the sum assured. This follows from the nature of the hypothesis; which supposing that one person will die every year till they are all extinct, reduces them at the extremity of life to one survivor, whose contribution must consequently be the sum assured.—It should, however, be observed, that the mean between the first and last payment is not to be understood as a simple *arithmetical* mean, or the *half* of the sum of those payments, but such a constant annual sum, which must so far exceed the value of an Assurance for the first year as to produce a *surplus* that, in consequence of being laid up, shall form a capital, which, together with its interest added to the annual payments, shall be sufficient to discharge the claims as they become payable; so that when the last life drops,

drops, just so much of the capital shall be left as, together with the interest due, shall be equal to one claim. Such a mean as here described may be found in Dr. Price's Treatise on Reversionary Payments, Question 10th *.

These are the chief principles from which the values of all Assurances on *single* lives are deduced. Those of the same kind on the *joint* continuance of the lives, or the *longest* of any number of lives, are found in a similar manner. I would only observe that as, in supposing any two events, the chance is universally *less* for the happening of *both* of them than of *one*, so the value of an Assurance on two joint lives which depends upon such a contingency must be *greater* than the value of an Assurance on the single life of one of them; for in the one case the claim is paid if they do not *both* live to the end of the term, that is, if *two* events do not happen; in the other case, it is paid if one does not live to the end of the term, or if *one* event does not happen. For the same reason the value of an Assurance for the longest of two, or any number of lives, will be *less* than on a single

* This is the same with the eighth Prob. in the third chapter of this work.

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life; for here the claim is not paid unless two or more events *do* happen, that is, unless two or more lives fail in the term. So that it is plain the greater the number of *joint* lives that are assured, the *greater* will be the value of the Assurance; and on the contrary, the greater the number that are assured during the *longest* liver, the *less* will be the value of the Assurance. The solutions of all cases of this kind being investigated from the same principles with those of single lives, they are consequently answered by the same rules in Dr. Price's Treatise, Questions 10th and 14th*.

Assurances on *Survivorships* are much more complicated, especially where several lives are concerned; but at present I shall only explain those of two lives, as they are by far the most common; and being properly understood, it will not be difficult to reason from the same principles in cases where a greater number of lives is concerned.—— To begin with the most simple, let us suppose the Assurance made only for one year on the contingency of one given life's surviving another. Let the age of the person to be survived be 50, and of the other 30;

* See Cor. to Prob. 8th, and Cor. to Prob. 9th, in the 3d chapter of this work.

that

that is, a given sum is to be paid in case the *former* should die *before* the latter in one year.—By the hypothesis*, the complement of a person aged 50 is 36, and that of a person aged 30 is 56. It follows, therefore, that the youngest person has 55 chances out of 56 for *living*, and the oldest person one chance out of 36 for *dying*, in the year. The payment of the claim depends upon either of two events; first, that they shall *both* die in the year and the youngest shall die last; secondly, that the oldest person *only* shall die in the year. The probability that they shall both † die is expressed by the two fractions $\frac{1}{36}$ and $\frac{1}{56}$ multiplied into each other, or $\frac{1}{36 \times 56}$. But that one in particular shall die *last* is an equal chance. *Half* of this fraction therefore, or $\frac{1}{2 \times 36 \times 56}$, will be the value of the expectation on the first of the two events. The second, which is by far the most considerable, will be expressed by the probability of the oldest per-

* See note, page 3.

† It is here taken for granted that the reader is so far acquainted with the doctrine of Chances as to know, that the probability that any number of independent events will all happen is the product of all their probabilities separately taken multiplied by one another.

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son's *dying*, or $\frac{1}{36}$, multiplied into the probability of the youngest person's *living*, or $\frac{55}{56}$. The sum of these two fractions therefore, or $\frac{1}{2 \times 36 \times 56}$ added to $\frac{55}{56 \times 56} = \frac{111}{4032}$, will express the *whole* ratio in which the sum proposed to be assured, after being discounted for a year, is to be diminished. By reasoning in the same manner, the value of any other Assurance of this kind may be determined.

If the term of the Assurance is extended to two, three, &c. years, or to the extremity of life, the principles by which the solution is investigated are exactly similar, provided the value is required in *one present payment*. Thus, the chance that the oldest shall die in any year is constantly $\frac{1}{36}$, and that the youngest shall die $\frac{1}{56}$; this appears by reasoning as in page 5. Half the fraction $\frac{1}{36 \times 56}$ will therefore always express the value of the expectation on the contingency of their both *dying* in the same year, and one in particular dying *last*. The other expectation after the first year depends upon the youngest person's *living* two, three, &c. years, and the oldest person's *dying* in the first, second, third, &c. years, and is determined

terminated by constantly multiplying $\frac{1}{36}$, expressing the latter probability, into the several fractions expressing the former probability. For the youngest person's living two years, there are, by the Hypothesis, 54 chances out of 56; for his living three years, 53 chances out of 56, and so on. The fractions $\frac{54}{56}$, $\frac{53}{56}$, &c. will therefore express the probability of his living two, three, &c. years. These being severally multiplied by $\frac{1}{36}$ and added to $\frac{1}{2 \times 36 \times 56}$, expressing the expectation on the first contingency, we shall have $\frac{109}{4032}$ for the ratio in which the sum to be assured, after being discounted for two years, is to be diminished in the second year; $\frac{107}{4032}$ for the ratio in which, after being discounted for three years, it is to be diminished in the third year, and so on. The sum, therefore, of all these terms is the whole value of the Assurance in one present payment. If it is only for a limited time, the number of terms must be equal to the number of years; if it is for life, the terms must be continued to the extremity of the oldest life. How these are summed up and reduced into general rules may be seen in *Simson's Select Exercises*,

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cises, Problem 32d—and Dr. *Price's* Treatise, Question 11th*.

In computing the values of Assurances on Survivorships in *annual* payments from the same principles with those on single lives, the reasoning is much more complicated. Suppose a given sum to be paid at the death of a person aged 61 should that happen before the death of another aged 36, and that 50 persons, all aged 61, had agreed to assure 100 *l.* each on their lives dependent on this contingency. By *De Moivre's* Hypothesis one will die out of 50 at the age of 36 in the first year, and two out of the same number at the age of 61.—One claim, therefore, will certainly become due in this year, towards which a contribution of 2 *l.* from each person must be paid.—That the other does not become due depends upon two events; first, that the youngest life which drops was one of those assured against either of the two oldest lives which failed in this year; secondly, that provided this was the case, the youngest person also died *before* the oldest one in the year. That the first event happens, there are two chances out of 50, or a probability which may be expressed

* See Problem 16th, chap. 3d, of this work.

by the fraction $\frac{2}{50}$, or $\frac{1}{25}$; that the second event happens is an *even* chance, which is expressed by $\frac{1}{2}$. These two fractions being multiplied into each other, we shall have $\frac{1}{2 \times 25} = \frac{1}{50}$ expressing the whole probability that the second claim does not become due; that is, $\frac{1}{50}$ th, or 2 *l.* must be deducted from this 100 *l.* and the remaining 98 *l.* divided into equal contributions, which being added to the former contribution of 2 *l.* will make the whole of the first year's Assurance 3 *l.* 19s. 2d; and as this is supposed payable at the beginning, and the claim not till the end of the year, it must be discounted for one year.—In the second year, there will be very nearly three Assurances less than in the first year; and as the deaths among the old and young lives continually bear the same proportion to each other, there will again be nearly two claims to be paid in this year; which being divided among fewer members, will render the contributions from each person so much the higher. In the third year there will be nearly six Assurances less than in the first, and three less than in the second year. The claims will, however, be much the same, and consequently the contributions

C higher

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higher than in the preceding year. By reasoning in the same manner, it will be found that the annual payments are continually increasing till they become greatest in the last year ; from whence it is obvious, that the true value of a *fixed* annual payment is determined by taking a mean, which will so far exceed the first yearly payment as to produce a *surplus*, that, with its accumulating interest, will be sufficient to discharge the claims of the last years. I have not attempted to ascertain with *exactness* how much the contributions increase and the claims diminish in every year, for the reasoning becomes so perplexed and complicated as to render it unfit for the present purpose. It appears, however, from this general view, that the contributions increase much faster than the claims diminish ; from whence a proof arises that the *last* payment is the most considerable, and that a *fixed* annual premium should be such a mean as I have described. The method of finding this mean in all cases may be seen in Dr. Price's Treatise, Question 11th.

The annual payments, both for the Assurances on Survivorships and Single Lives, are generally determined by dividing the value of the Assurance in one payment by

the value of an annuity on the joint or single lives; and the reason of this may be easily inferred.—Thus; let the single premium of Assurance on the above Survivorship be 50 *l*. If, instead of one payment, it is agreed to take a fixed *annual* payment during the continuance of the joint lives, it is plain the value of this will depend on the number of years they will exist; that is, on the number of years which, one with another, all joint lives at those ages do exist together, some existing as much *longer*, as others exist a *shorter* time; or, in other words, on the *mean* duration of the joint continuance of the lives, called by *De Moivre*, and other writers on the subject, the *expectation* of the joint lives. In the present instance, this expectation is just equal to ten years and a half; so that the annual payment will be 4 *l*. 15 *s*. 3 *d*.; for ten payments and a half, of 4 *l*. 15 *s*. 3 *d*. each, will be equal to 50 *l*, supposing the person assured to make no interest of his money; but as the contrary is always understood, it follows that 50 *l*. received in this manner, cannot be an equivalent to the same sum received in one present payment; for the latter, in $10\frac{1}{2}$ years, at 4 per cent. compound interest, will accumulate to

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75*l.* 9*s.* 6*d.* whereas the former, under the same circumstances, will not exceed 60*l.* 13*s.* 4*d.*—The preceding annual payments, therefore, should be increased; and the true value of an annuity on the joint lives being nearly $7\frac{1}{2}$ (allowing a discount at 4 per cent.) this, and not $10\frac{1}{2}$, will be the sum by which the *single* premium of 50*l.* must be divided, in order to find the *annual* premium equivalent to it*. A similar reasoning may be applied to Survivorships *for terms*, and to Assurances on Single Lives *for terms* and *continuance*.

In examining the principles upon which the calculations of Assurances are founded, I have generally chosen to make use of *De Moivre's* Hypothesis, as being the most simple. The same reasoning, however, may be formed from any table of observations; many of which correspond very nearly with the hypothesis.

* This calculation supposes the first payment to be made at the *end* of the year; but if it is made at the *beginning* of the year (which is commonly the case) unity must be added to the value of the joint lives. This, in the present instance, would reduce the annual premium to 5*l.* 17*s.* 8*d.* which, without the addition of unity, is 6*l.* 13*s.* 3*d.*

S E C T.

S E C T. II.

On the different Methods of determining the State of a Society, whose Business consists in making Assurances on Lives.

HAVING considered the principles from which the values of Assurances in general are investigated, let us now suppose a society established for making such Assurances. It is obvious, that so far as it does not regulate itself according to these principles, it must proceed in the dark, and be either requiring exorbitant contributions on the one hand, or laying the foundation for future disappointment on the other. It becomes, therefore, of the last consequence that a society of this kind should always be able to determine its real state, and I will beg leave to give the following methods for this purpose.

I. Let an exact account be taken of the proportion of assured persons who have died annually, or of the decrements of life at the different ages of the members, from the establishment of the society to the present time. If these decrements of life should appear to be *less* than those in the tables

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from which the calculations are made, a demonstration will arise of the thriving state of the society, and it will enjoy advantages equivalent to the excess of the decrements of life in the tables above those in the society.—If, on the contrary, these decrements of life should appear to be greater than those in the tables, or that a larger number dies than the calculations suppose, the society will sustain losses proportionable to this difference.—The probabilities of life in a society should be always a little *higher* than those in the tables, in order to balance the necessary expences of management, and to enable it to lay up against any particular season of mortality; from whence it follows that, should the probabilities of life in the society be *lower* than those in the tables, its permanency must be soon destroyed, if others are not used which make the decrements of life more considerable.

II. The shortest method of trying the state of the society (as far as it assures *single* lives) is by comparing the amount of the *annual payments* which have been made for such assurances, with the amount of the annual *claims* derived from them; and the following general rule may be given for this purpose :

“ Find

“ Find the sum of all the annual pay-
 “ ments for assurances on the *whole dura-*
 “ *tion* of single lives. Find also the sum
 “ of all the claims grounded upon these
 “ payments. In like manner, find the sum
 “ of all the annual payments for assurances
 “ on single lives for any term not exceeding
 “ seven years. Find also the sum of all the
 “ claims derived from these last payments.
 “ If in the *latter* case, the sum of all the
 “ payments is equal to the sum of all the
 “ claims, and in the *former* case, *greater*
 “ in the proportion of three to two, it will
 “ appear that the payments are sufficient to
 “ support the engagements of the society,
 “ supposing no expences of management;
 “ and as much as the payments are in a
 “ higher proportion than this to the claims,
 “ so much profit will the society enjoy.”

The principles from which this rule is
 derived will appear from the following ob-
 servations.—Suppose a body of people all
 40 years of age to form themselves into a
 society for the purpose of assuring 100*l.* each
 on their lives for their whole duration.
 Suppose also that they keep up the society to
 the same constant number by admitting con-
 tinually new members at the age of 40 in
 the room of such old members as die off.

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It is plain in these circumstances that the whole collective body of members will be continually growing older till their mean age comes to be as great as possible. To this greatest mean age they will attain in a number of years equal to the difference between the age at admission, or 40, and the extremity of life, that is in 46 years, supposing the extremity of life to be 86. During this period the number of members dying annually will be continually increasing, till at the end of it about twice as many will die annually as died at the first establishment of the society. The claims, consequently, will also increase annually till they are doubled.* Had, therefore, the payments necessary to support the Assurances been made from year to year in the manner described in page 8, they must have been always increasing till they became double; but if the value of the Assurance is paid by contributing an annual sum constantly the same, this annual sum must be a mean between the payment due the first year and the double payment which would become necessary at the end of 46 years, agreeable to what has been observed in page 19.—According to *De Moivre's Hypothesis* * the

* See note, page 3.

value

value of an Assurance of 100 *l.* for the whole duration of a life aged 40 is in annual payments (the first to be made immediately) 3 *l.* 4 *s.* reckoning interest at 4 per cent. The value by the same Hypothesis of the same Assurance for a single year is 2 *l.* 1 *s.* 10 *d.* At first therefore in the society there will be an annual *surplus* equal to 1 *l.* 2 *s.* 2 *d.* from each member over and above the growing interest of his payments. This surplus will decrease continually as the society grows older, and in 46 years it will vanish; but it will be sufficient in that time to raise a capital, the annual interest of which added to 3 *l.* 4 *s.* from each member will make up a sum equal to that to which the annual claims will increase in 46 years. Let the number of members be 1000, all when admitted 40 years of age, and having 100 *l.* each assured to them for the whole duration of their lives at a *fixed* annual payment. By *De Moivre's* Hypothesis the number of deaths the first year will be 21.74. Each of those deaths will produce a claim of 100 *l.* and therefore the society will at the end of the year have the sum of 2174 *l.* to pay; but the contributions of 3 *l.* 4 *s.* each will amount to 3200 *l.* and 128 *l.* being a year's interest-

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interest for 3200 *l.* the total, or 3328 *l.* will be the whole income of the society the first year. The difference between this sum and 2174 *l.* (or 1154 *l.*) will be the first year's surplus to be laid up and improved at compound interest.—A surplus a little *less* will be left the second year, because more will die that year. Another surplus still less will be left the third year for the same reason, and so on for 46 years. The annual claims having been all this time in a state of increase, they will at last come to be 43 $\frac{1}{2}$. The annual sum then payable by the society will be 4350 *l.* or 1150 *l.* *more* than the annual contributions, but at that period the capital formed from the annual surplusses will have increased to such a sum as at 4 per cent. will yield this interest; that is, it will have increased to the sum of 28,750 *l.*

Now it should be observed in the preceding account, that 3200 *l.* the amount of the contributions of 1000 members all 40 at admission, is to the amount of the claims almost exactly as three to two. In some ages this proportion may be found to be lower than three to two; in others it may be found higher; the rule therefore which has been given may be considered as sufficiently

ciently exact when applied to the circumstances of a society which assures lives at all ages. If this society has been of any standing, it affords a test rather too severe; for we have seen that the proportion of the annual payments to the annual claims must be continually decreasing till in a course of years the claims become *equal* to the payments, and afterwards *greater*. But during the first eight or ten years from the commencement of the society this rule will be pretty near the truth, and as it errs after this time on the safe side, no bad consequences can arise from having recourse to it.

—From what has been said, it is, however, easy to determine with exactness in any period what proportion the claims should bear to the annual payments; for in 23 years they will become equal, and in 46 years the claims will exceed the payments in the proportion of 4350 *l.* to 3200 *l.* See page 26.—As to Assurances for terms not exceeding seven years, I have observed that the claims derived from them ought in all periods of the society to be nearly equal to the payments. In order to prove this, let us take as before a life at the age of 40. 100 *l.* assured on such a life for a year is worth, I have said, in present payment,

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2*l.* 1*s.* 10*d.* From a thousand such Assurances will arise at the end of the year 21.739 claims of 100*l.* The society therefore at the end of the year will have a debt to pay of 2173*l.* 18*s.* but the payments at the beginning of the year at 2*l.* 1*s.* 10*d.* each amounted only to 2090*l.* 6*s.* In the case therefore of Assurances for a single year the claims ought a little to exceed the payments; and the reason is, that the payments being made at the beginning of the year, they are supposed in the calculation to bear a year's interest, which is to be added to them in order to make up a sum equal to the claims at the end of the year.—Again 100*l.* assured on the same life for two years is worth 2*l.* 2*s.* 3*d.* in two equal payments, the first to be made immediately and the second a year hence. 1000 such payments the first year will make 2113*l.* but the claims, it has just appeared, will amount to 2173*l.* 18*s.* and the second year they will be a little more; but (the number of such Assurances being supposed to be kept up) the payments will be the same. In the case therefore of Assurances on single lives for two years the payments will both years fall short of the claims. An Assurance on the same life for three years is worth
2*l.*

2*l.* 2*s.* 9*d.* and 1000 such payments the first year will make 2136*l.* but the claims in this year will amount to 2173*l.* 18*s.* the second year to a little more, and the third to still more.—An Assurance on the same life for four years is worth 2*l.* 3*s.* 2*d.* 1000 such payments the first year will make 2158*l.* which still fall short of the claims in this year, and of course must fall short of them in the three succeeding years. In an Assurance for five years the annual payments will be 2*l.* 3*s.* 7*d.* 1000 such payments will make 2181*l.* which sum exceeds the claims in the first year but will be less in the four following years. In Assurances for six years the payments will exceed the claims the first year, be nearly equal to them the second year, and less the three succeeding years. The like is true in Assurances for seven years; and what has been now applied to the age of 40 may, without any great difference, be applied to any other age. Take therefore all Assurances together made at different times in the society, some for one, some for two, three, &c. years, it may be expected that the claims and payments will be nearly equal, and if the latter exceed the former, the society must be so far a gainer.

III. The

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III. The third and last method is the most decisive, and determines not only whether the society is in a thriving state, but likewise of what stock it is possessed after paying to all the different members the value of their interests in their respective Assurances. It may be inferred from the second method (supposing the Assurances paid for at the beginning of the year) that the stock which should be raised from Assurances on single lives for *terms* from one to seven years is nearly one year's payment for those Assurances; and it has appeared also that from the payments of a standing body of 1000 members assuring 100 *l.* each and all aged 40 at admission, the capital which should be raised in 46 years is 28,750 *l.* supposing interest at 4 per cent. and the probabilities of life as they are according to *De Moivre's* Hypothesis. By applying these rules, therefore, under proper restrictions, the surplus stock of the society may be nearly ascertained so far as relates to Assurances on *single lives*; but this method does not extend to Assurances on the *joint* lives, the *longest* of two lives, or on *survivorships*. These are indeed in common with all the other Assurances included in the first method, but as it can never be proved from
hence

hence what the society is really possessed of after fulfilling its engagements, and as it will be proper that an institution of this consequence be examined by every possible test to determine its stability, I shall beg leave to lay down the following rule for this purpose.—“ Find the *present* value in *single* “ payments of all the Assurances subsisting “ in the society.—Find likewise so much “ of all the different annual payments as is “ proportionable to that part of the year “ which is yet to come, before the next “ payments become due. Add both these “ values together, and let their sum be reserved.—Find next the *present* values in “ *single* payments of all the *annual* premiums due upon the several Assurances; “ to these add the present stock or capital “ of the society; and as far as this last “ sum *exceeds* the sum reserved above, of so “ much *surplus* stock will the society be “ possessed; but if it falls short of this reserved sum, a demonstration will arise “ that the plan of the society is inadequate.”

EXAMPLE.

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E X A M P L E.

Suppose there are 1000 Assurances for 100 *l.* each on the whole duration of life, and that the annual premiums are 3 *l.* 4 *s.* each, or in the whole 3200 *l.*—Let these members be now 50 years of age; and the value of an Assurance in one present payment on such a life being by *De Moivre's* Hypothesis and at 4 per cent. 52 *l.* 10 *s.* 5 $\frac{1}{2}$ *d.* 1000 such Assurances will be worth 52,523 *l.* Suppose half a year has elapsed since the last annual payments were made, half these therefore, or 1600 *l.* must be added to the preceding sum, which will increase it to 54,123 *l.*—The value of an Annuity on a life of 50 at 4 per cent. is worth 11 $\frac{1}{3}$ year's purchase nearly, and as the annual payment is to be made in half a year, this value must be increased by half-unity, which will make it equal to 11 $\frac{2}{3}$.—If 3200 *l.* be multiplied by 11 $\frac{2}{3}$ we shall have 37866 *l.* for the present value of the annual premiums in one payment.—In the same manner let there be 500 Assurances of 100 *l.* each for a term of years, of which two are unexpired, at 1095 *l.* per ann.—Let the age of these lives be 43 years, and the present value in

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one payment of those Assurances on their lives for two years is 2199 *l.*—Suppose their annual payments just due, nothing in this case is therefore to be added.—The value of a life of 43 for one year is worth $\frac{939}{1000}$ year's purchase. If 1095 *l.* be multiplied by this fraction, and the annual payment just due be added to the product, we shall have 2123 *l.* for the whole value of the premiums in one present payment.—Again. Let there be 20 Assurances of 100 *l.* each on the joint continuance of two lives aged 45 and 55 at the annual premiums collectively of 109 *l.* 16 *s.*—The value of those Assurances in one *present* payment will be 1304 *l.* 6 *s.** Suppose the premiums to have been paid three quarters of a year ago; one quarter of them, or 27 *l.* 9 *s.*, must be added to the preceding value which will make the whole 1331 *l.* 15 *s.* By multiplying the annual premiums by the value of the joint lives, increased in this case by three fourths of unity, as the next payment is to be made in a quarter of a year, we shall have 965 *l.* 12 *s.* for the present value of those premiums in

* The method of finding these values, and in general the values of all Assurances on Lives and Survivorships for terms and perpetuity, will be particularly explained in the third chapter.

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one payment.—In the same manner may the Assurances on the longest of two lives be found. Let there be ten such Assurances for 100 *l.* each on two lives whose respective ages are 50 and 60, at the annual premiums collectively of 18 *l.* 10 *s.* Supposing their payments just made, the whole year's premium is here to be added to the present values of those Assurances, which will be found equal to 469 *l.* 12 *s.*—The premiums multiplied by $13\frac{1}{4}$ (or the value of an annuity on the longest of two lives aged 50 and 60) will produce 245 *l.* 2 *s.* for the value of those premiums in one present payment.—Lastly, let there be 60 Assurances of 100 *l.* each on the survivorship of one life beyond another. Let the present age of the persons to survive be 50, and of the persons to be survived 60, and let the annual premiums collectively be 171 *l.*—Suppose one quarter of a year to have elapsed since the last premiums were paid. 128 *l.* 5 *s.* is therefore to be added to 2578 *l.* 16 *s.* which by the rules in chap. 3d may be found to be the present values of those Assurances in one payment, and the total will be 2707 *l.* 1 *s.* The premiums multiplied by $7\frac{1}{4}$, or the value of two joint lives whose respective ages are 50 and 60 increased by one fourth of unity, will

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will produce 1248*l.* 6*s.* for the value of those premiums in one present payment. Now the values of all these Assurances and proportional parts being added together will amount to 60,830*l.* 8*s.*—The values of all the annual payments due for these Assurances being also added together will amount to 42,448*l.* If 18,382*l.* 8*s.*, the difference between those two sums, be not found in the stock of the society, or, which is the same thing, if the stock of the society added to 42,448*l.* be not equal to 60,830*l.* 8*s.*, a proof will ensue that the society is not able to fulfil its engagements. As much therefore as the two former sums exceed the latter, of so much *surplus* stock will the society be possessed.

Having been so particular in this example, and the principles of those calculations being so obvious, any further explanation of them can hardly be necessary *. I am sensible

* Suppose the society had engaged ten years ago to pay 100*l.* at the death of a person aged 40. This person being now 50, the value of such a sum in *present* money is 52*l.* 10*s.* 5*d.* which would therefore express the value of his present interest in the reversion, supposing him to have completed his payment for it. But if he has agreed to pay for it by an annual premium of 3*l.* 4*s.* during life, it will follow that he *owes* as much as such an annual premium dependent on his life is worth.—A

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fible that a work of this kind would be attended with great labour where the society is numerous; for the values of all the different Assurances that do not agree in time, date, and in the ages of the persons assured, must be calculated *separately*; but no labour should be avoided that is proper to throw light on the affairs of such a society, and to give a more clear and satisfactory view of its state.

I shall conclude this section with observing that the state of the *Equitable Society* (whose business consists in making all the different assurances I have described, and who

life of 50 is worth $11\frac{1}{3}$ year's purchase, or $12\frac{1}{3}$ as the first payment is to be made immediately, the present value, therefore, of what this person is indebted will be $3l. 4s.$ multiplied by $12\frac{1}{3}$ or $39l. 7s.$ Consequently his interest, or the balance in his favour will be no more than the difference between $52l. 10s. 5d.$ and $39l. 7s.$ or $13l. 3s. 5d.$ But if he applies for this balance at any other time than when his annual payment becomes due; it is plain, since the values of lives in the table suppose the first payment not to be made till the end of a year, that the annual premium must be multiplied by $11\frac{1}{3}$ increased by as much less than unity as what remains of the year is less than the whole year. By proceeding in the same manner with all Assurances for Terms, Survivorships, &c. in the society and adding those differences together, we have the whole of what it is indebted to its members; and so far as this falls short of, or exceeds its *stock*, so far will the society be in a precarious, or in a flourishing condition.

calculate their premiums from the principles explained in section I.) has lately been examined by each of the methods here proposed, and I shall now lay the result of those examinations before the public.

By the first method it has been found that during the term of nine years (from January 1768 to January 1777) the proportion of deaths in the Society to those in the Table of Observations for London, from which its premiums have been calculated, are,

From the age of 20 to 30 as	7 to 17
30 to 40 as	3 to 5
40 to 50 as	1 to 2
50 to 60 as	2 to 3
60 to 70 as	14 to 11

Or in all ages from 20 to 70 as two to three nearly; that is, where the tables have supposed three claims would have happened, only two have taken place in the society.—

By comparing the decrements of life in this society with those in Dr. *Halley's* table, which agree very nearly with *De Moivre's* Hypothesis, they are found to be,

From the age of 20 to 30 as	5 to 9
30 to 40 as	5 to 6
40 to 50 as	6 to 7
50 to 60 as	5 to 6
60 to 70 as	5 to 3

D 3

Or

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Or in all ages from 20 to 70 as 11 to 13 nearly.

By the *second* method, or by determining the proportion of the claims to the premiums for Assurances on Single Lives, it appears that the claims in this society have on an average for these last nine years been 3115 *l. per ann. less* than they should have been; that is, the Society has enjoyed a clear *annual* profit during this term of 3115 *l.* from that part of its business only which consists in assuring Single Lives.

By the *third* method, which is the most certain and decisive, the surplus stock of this Society is proved by calculating even by the *London Table*, to be considerably above 30,000 *l.*—In consequence of those concurring testimonies of the flourishing state of this Society it was resolved in June 1777 to reduce the premiums of Assurance one *tenth*, and to return to every member the excess of the payments he has actually made above the payments he would have made, had the reduction taken place from the time of his admission.—Those reductions and returns ought undoubtedly to be considered as a diminution both of the annual profits and surplus stock of the Society; and it may also be observed that if the calculations

culations had been made *after* the premiums were reduced, their present values would have been *less*, which by increasing the difference between them and the present values of the Assurances in *one* payment, would have likewise proportionably diminished the surplus. But these objections have very little weight.—The annual profits of the Society considerably exceed the reduction, together with all the expences of management; nor are the returns greater than the excess of the surplus stock above 30,000 *l.* Had also the present values of the *single* and *annual* payments been calculated by Dr. *Halley's* Table (as it is evident they should have been*,) the former would have *decreased*, and the latter *increased* in a much larger proportion than one tenth.—The accumulating interest, therefore, on a surplus stock of 30,000 *l.* together with the annual profits which still remain over and above the expences of management, perfectly secure this Society from any future danger, and render its late resolutions not only *safe*, but highly *just* and *necessary*.

* See page 37.

C H A P. II.

An Introduction to the Doctrine of Annuities.

S E C T. I.

Of Annuities on Lives.

THE value of a *Life Annuity*, like that of an *Assurance*, is more or less easily determined in proportion to the number of lives, on the continuance of which the annuity depends. Where, therefore, only *one* life is concerned the operation will be most simple, and if the Annuity is limited to a single year it becomes the exact converse of that for an *Assurance* during the same time ; since in the former, a given sum is to be paid if the life *exists* one year, in the latter the sum is to be paid if the life *fails* in the year ; so that if it were an *even* chance whether it would exist through a year or not, the value of an Assurance and Annuity would be equal : and in proportion as the
 chance

chance of *living* exceeds that of *dying* in the year, in the same proportion will the value of the *Annuity* exceed the value of the *Affurance*.—Suppose, for example, that A. and his heirs were entitled to 50 Annuities of 100 *l.* each on the single lives of as many persons, whose common age was 39; that is, that they were to receive 100 *l.* from every one of 50 persons of this age at the end of every year while they lived. By Dr. *Halley's* Table of Observations it appears that 49 out of the 50 will live one year; the value, therefore, of the first payments of those annuities is 100 *l.* multiplied into 49, (or 5000 *l.* the sum of all the payments, lessened in the ratio of 49 to 50) which is equal to 4900 *l.* But as this money is not payable till the *end* of the year, it must be discounted for a year, so that the present value together with the interest may in this time just accumulate to 4900 *l.*—From hence it follows that the value of the first payment of an Annuity in the present instance exceeds that of an Assurance for a year in the ratio of 49 to 1; for in page 2 it is proved that the premium for assuring those several sums on 50 such lives is only 100 *l.* discounted for a year.—Had the Annuity been upon 50 single lives whose com-
mon

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mon age was 59, two persons would have died by Dr. *Halley's* table in the year, so that the first payment would have been 100 *l.* multiplied into 48 (or 5000 *l.* lessened in the ratio of 48 to 50) which is equal to 4800 *l.* and in page 2 it is proved that the premium for assuring those sums is 200 *l.**; hence the first payment of an *Annuity* on a life of 59 exceeds an *Assurance* for a year on the same life only in the ratio of 48 to 2 or 24 to 1.—At the age of 89 the number of those that die *exceeds* the number of those that live one year; in which case the value of the first payment of an *Annuity* becomes *less* than the value of an *Assurance* for a year of a sum equal to the *Annuity*. It may be observed from those calculations, that the sum of the values of the first payment of an *Annuity* and *Assurance* for one year is always equal to the value of a sum equal to the *Annuity* to be received *certainly* at the end of the year. This indeed is self-evident, for if a given sum is to be received whether a life *does* or *does not* exist to the end of a year, one or other of these two events must happen, and consequently the expect-

* It should be observed that both these sums of 4800 *l.* and 200 *l.* are to be discounted for a year.

tation on *both* of them is the expectation of a certainty.

The value of the different payments of a life annuity for two, three, or any number of years is determined much in the same manner.—Suppose the age of the person to be 66, and the probabilities of life to decrease according to *De Moivre's* hypothesis*, the chances in this case for the annuitant's living to the end of the first year are 19 out of 20, and the value, therefore, of 1 *l.* to be received at the end of a year on the contingency of his living so long will be expressed by the sum discounted for a year and multiplied into the fraction $\frac{19}{20}$.—In the second year the chances for his living to the end of that year are only 18 out of 20, so that the value of 1 *l.* to be received at the end of this year will be expressed by the sum discounted for two years and multiplied into the fraction $\frac{18}{20}$.—The value of 1 *l.* to be received at the end of the third year, by reasoning in the same manner, will be expressed by the sum discounted for three years and multiplied into the fraction $\frac{17}{20}$.—By proceeding thus through any number of

* See this hypothesis explained in note, page 3.

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terms and adding the several products together, we shall have the value of an Annuity of 1 *l.* upon this life for a number of *years* equal to the number of terms; and if we continue the operations for 20 terms, it is evident that the numerator of the 20th term vanishes, or, which is the same thing, that the life in 20 years beomes extinct; and consequently that the sum of all the preceding 19 terms will express the value of an annuity upon the *whole life* of a person whose age is 66.—This reasoning may be extended to all ages, as well as to any table of observations.—Let the age, for instance, be 20, and the probabilities of life as in Dr. *Halley's* table., It appears from thence that out of 598 persons at the beginning of the first year, no more than 592 will live to the beginning of the second year, 586 to the beginning of the third year, 579 to the beginning of the fourth year, and so on. The value, therefore, of the first payment of an annuity on a life whose age is 20 is expressed by the sum discounted for a year and multiplied into the fraction $\frac{592}{598}$. The value of the payment for the second year is expressed by the sum discounted for two years and multiplied into the fraction $\frac{586}{598}$.

The

The value of the payment for the third year is expressed by the sum discounted for three years and multiplied into the fraction $\frac{579}{598}$.

—If the operations are continued to 71 terms it is evident from the table that the numerator of the fraction which constitutes the 71st term vanishes; and consequently the sum of the different products for 70 terms will express the value of the Annuity for the whole duration of a life aged 20.

This subject may be otherwise explained in the following manner. —Suppose a Life-Annuity of 1 *l.* payable yearly to every one of 515 persons all now aged 20, the first payment of which is to be made a year hence. It appears from the *Northampton* Table of Observations * that only 507 of these persons will be living at the end of a year, and consequently that the money to be then paid will be only 507 *l.* The present value, therefore, of the first payment of the Annuities will be the sum which now put out to interest will increase in a year to 507 *l.* that is, it is 507 *l.* discounted for a year, or 487 *l.* 10 *s.*; for this sum added to its interest for a year (reckoned at 4 per cent.) will just make up 507 *l.*—From the same table

* See table 3d, Appendix.

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it appears further that of 515 persons living at 20 years of age only 499 will be living at the end of two years. The present value, therefore, of the second payment of the Annuities will be the sum which being now put out to compound interest at 4 *per cent.* will increase in *two* years to 499 *l.*—This sum is 461 *l.* 7 *s.* In like manner 491, 483, 475, &c. being the number living at the end of 3, 4, 5, &c. years, the value of the 3d, 4th, 5th, &c. payments of the Annuities will be 491 *l.* 483 *l.* 475 *l.* &c. discounted for 3, 4, 5, &c. years respectively, and continued to the year in which all the lives become extinct.—The total of all these values is 8204 *l.* which therefore is the sum that would be sufficient, if improved at 4 *per cent.* to make good the payment of an Annuity of 1 *l.* for life to every one of 515 persons aged 20, according to the *Northampton* Table of Observations. The value, therefore, of such an Annuity payable to only one such person must be the 515th part of 8204 *l.* or 15 *l.* 18 *s.* 6 *d.*—This seems to be all extremely clear; but he that would see it drawn out at length and fully explained should consult a book entitled *Calculations deduced from first Principles, &c.* *

* The author of this work (Mr. Dale) has lately published a SUPPLEMENT to it; and he deserves the highest

Supposing the probabilities of life to decrease in arithmetic progression agreeable to *De Moivre's* hypothesis, the sum of the several products is easily calculated by *Quest. 56, vol. 2, Dodson's Mathematical Repository*, or by *Prob. 1st, Cor. 5th, Simpson's Doctrine of Annuities*, or by *Prob. 2d* in the next chapter.—But those rules cannot be applied to *tables of observation* with sufficient accuracy, nor is it possible that any general rules, *strictly true*, can be delivered for this purpose. Under these circumstances, therefore, it becomes necessary to

highest praise for the pains he has taken to detect and expose those bubbles, which were some time ago established under the name of *Societies for the Benefit of Old Age*. Most of these societies, convinced of their mistake, have dissolved themselves. But there is one still left, on which none of the calls of justice and humanity have been able to make any proper impression. I mean the *Laudable Society of Annuity for the Benefit of old Age*, whose office is held at the bottom of *Bartholomew Lane*. In opposition to the plainest evidence, this society goes on to offer *double* the Annuity it can afford to pay; and the late transactions in it (as related by *Mr. Dale* in his *Supplement* just mentioned) exhibit an instance of such an obstinate and wilful perseverance in imposition, as hath seldom been equalled.—I am sorry to add, that this censure is applicable to another Annuity society called also LAUDABLE, but in reality PERNICIOUS, as many suffering widows will some time or other experience.

proceed

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proceed through every fraction to the extremity of life, which in younger ages is attended with vast labour.—These difficulties, however, may in a great measure be obviated; and I shall in the subsequent section lay down a method by which the values of Annuities upon lives of all ages may be computed nearly in the same length of time with a *single* Annuity upon a life of the youngest age.

The values of Annuities upon the *joint* continuance of two lives are determined from the same principles with those on single lives.—Suppose the ages of two persons were 50 and 60. Their complements by the hypothesis (that is, the difference between their ages and 86 *) will be 36 and 26, and the chances for their living singly the first year will be 35 out of 36, and 25 out of 26; consequently the probabilities of their *both* living to the end of this year will be expressed by the two fractions $\frac{35}{36}$ and $\frac{25}{26}$ multiplied into each other †. In the same manner the probabilities of their *both* living to the end of the second year will be expressed by the two fractions $\frac{34}{36}$ and $\frac{24}{26}$ mul-

* See note, page 3.

† See note, page 13.

tiplied

tiplied into each other. The probabilities of their *both* living to the end of the third year by the two fractions $\frac{33}{36}$ and $\frac{23}{26}$ multiplied into each other, and so on for 26 years; at the end of which the numerator of the fraction expressing the probability of the oldest life's existence vanishes. Those several fractions, therefore, multiplied into the sum properly discounted, and then added together, will give the whole value of the Annuity upon the *joint* continuance of two such lives.

The values of Annuities on the *joint* continuance of three or any other number of lives are found in the same manner; the several fractions expressing their respective probabilities each year being multiplied into one another, and also into the sum discounted as above, and the aggregate of the products arising from those multiplications continued to the extremity of the *oldest* life will give the whole value of the Annuity.

It is plain from what has been observed that the value of an Annuity upon a *single* life must be always greater than that of an Annuity upon *two joint* lives of the same age; for in the first, the money is to be received if only *one* event happens, in the se-

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cond it is not to be received unless that event and another both happen.—For the same reason an Annuity upon *two joint* lives is worth more than an Annuity upon *three joint* lives of the same age, inasmuch as the money in the one case is to be received if only *two* events happen, in the other it is not to be received unless those two events and one more all happen. Universally therefore, the *greater* the number of lives the *less* will be the value of the Annuity on their *joint* continuance; and this number may be increased so far as even to render such an Annuity of *no* value.—In the case of an equal decrement of life, that is, supposing that out of any number living at any given age an equal number will die every year till all are dead, the values of those Annuities for two or three joint lives may be calculated by Quest. 64 and 69, vol. 2. of *Dodson's* Mathematical Repository, and by Prob. 1, Cor. 5, of *Simpson's* Doctrine of Annuities, or for two joint lives by Prob. 3. in the next chapter.—When the calculations are to be made from a table of observations these rules are of no use, and the *real* values can only be determined by a tedious multiplication of all the terms.—This requiring too much labour, mathematicians

maticians have generally contented themselves with approximations to the true values.—Mr. *Simpson* in his *Select Exercises*, Prob. 9th *, has given a rule for determining the value of three joint lives from the given value of two joint lives with tolerable precision. But it is evident that the more accurate the *latter* value is given, the nearer an approximation will Mr. *Simpson's* rule be to the real value of the *former*.—On this and many other accounts, tables of the *exact* values of two joint lives are exceedingly desirable; and should any person undertake such an useful and necessary work, he may perform it with a vast abridgment of labour by the method laid down in the following section.

With respect to Annuities on the *longest* of any number of lives, the reasoning is not so simple; and if there are more than two lives concerned, it becomes so complicated as not to be capable of being explained in a few words. I shall, therefore, confine myself to the single case of two lives, which being properly understood, the same principles may be applied without

* This is the same with the 5th Problem of the following chapter.

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much difficulty to any greater number of lives.

Suppose the ages of the two persons are 50 and 60, and, for the sake of more perspicuity, that the decrements of life are according to *De Moivre's* hypothesis*.—The chance of the youngest person's living one year is expressed by $\frac{35}{36}$, and the same chance with regard to the oldest by $\frac{25}{26}$. The chance of the youngest person's *not* living one year is $\frac{35}{36}$ subtracted from unity †, or $1 - \frac{35}{36}$, and the same chance with regard to the oldest is also $1 - \frac{25}{26}$. These two expressions being multiplied into each other will give the probability of their *both* dying

* See note, page 3.

† A certainty is always denoted by *unity*; and, therefore, the *difference* between the probability of an event's *happening* and this certainty, must express the probability of the same event's *not* happening.

It should be farther remembered here, that the probability that any two independent events will both happen is always the probability that the one will happen multiplied by the probability that the other will happen. For more particular information on this subject the reader is referred to *Simpson's* and *De Moivre's* Doctrine of Chances, and also to the 23d and following Questions in the 2d vol. of *Dodson's* Mathematical Repository.

in the year, which is equal to $1 - \frac{35}{36} - \frac{25}{26} + \frac{35 \times 25}{26 \times 36}$. For suppose the first fraction $1 - \frac{35}{36}$ to be multiplied into unity, then it will still remain equal to $1 - \frac{35}{36}$. Let it again be multiplied into $\frac{25}{26}$ and it will become equal to $\frac{25}{26} - \frac{25 \times 35}{26 \times 36}$. But $\frac{25}{26}$ was a *negative* quantity, therefore the signs of the two last fractions must be changed, or wrote down $\frac{25}{26} + \frac{25 \times 35}{26 \times 36}$. These being added to $1 - \frac{35}{36}$, will give $1 - \frac{35}{36} - \frac{25}{26} + \frac{25 \times 35}{26 \times 36}$ for the whole product of $1 - \frac{35}{36}$ into $1 - \frac{25}{26}$.—Now since those expressions give the probability that the two lives shall *both* fail in the year, it follows that if they are subtracted from unity*, they will give the probability that they shall *not* both fail in the year; that is, that one of them at least will live to the end of this year. In order to subtract $1 - \frac{35}{36} - \frac{25}{26} + \frac{25 \times 35}{26 \times 36}$ from unity it is evident that nothing more is necessary than to change the signs of this expression and then add it to unity; in which case we shall have $1 - 1 + \frac{35}{36} + \frac{25}{26} - \frac{25 \times 35}{26 \times 36} = \frac{35}{36} + \frac{25}{26} - \frac{25 \times 35}{26 \times 36}$ for the true value of

* See note, page 52.

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the above probability.—In the same manner the chance of one of them at least living to the end of the second year is expressed by $\frac{31}{36} + \frac{24}{26} - \frac{24 \times 31}{26 \times 36}$.—The chance of one of them at least living to the end of the third year by $\frac{33}{36} + \frac{23}{26} - \frac{23 \times 33}{26 \times 36}$, and so on for a number of years equal to the complement of the youngest life (or the difference between its present age and 86,) as it is plain that *all* the fractions will not vanish sooner.—If those several expressions are respectively multiplied into the sum discounted for 1, 2, 3, &c. years, and the products are added together, we shall have the whole value of the Annuity for the continuance of the longest of two lives whose ages are 50 and 60.—By reasoning in the same manner the value of an Annuity on the longest of any two lives, and by any table of observations, may be determined. But if the values of the single and joint lives are given, the value of an Annuity during the continuance of the *longest* of them is immediately found; for it appears from what has been said on single and joint lives, pag. 43 and 48, that $\frac{35}{36}$ multiplied into 1 *l.* discounted for a year expresses the value of the first payment of

an Annuity on a life whose age is 50, that $\frac{25}{26}$ multiplied into the same sum expresses the like value for a life of 60, and that $\frac{25 \times 35}{26 \times 36}$ multiplied as above expresses the value of the first payment of an Annuity on the joint continuance of those lives. It follows, therefore, that $\frac{35}{36} + \frac{25}{26} - \frac{25 \times 35}{26 \times 36}$ multiplied into 1 *l.* discounted for a year, and expressing the value of an Annuity on the *longest* of two lives for one year, is equal to the difference between the value of the *joint* lives and the *sum* of the values of the *single* lives for the same term.—The like may be observed with respect to the second, third, &c. years; from whence we have the following general rule:—" Subtract the value
 " of an Annuity on the *joint* lives from the
 " sum of the values of an Annuity on the
 " *single* lives, and the *remainder* will give
 " the value of an Annuity on the continu-
 " ance of the *longest* of two such lives *."

If

* **EXAMPLE.** Suppose the value of an Annuity is required on the longest of two lives whose ages are 40 and 50.—By Table 5th in the Appendix, the value of a single life of 40 is 13.196, and by the same Table the value of a single life of 50 is 11.344.—Their *sum* therefore is 24.54, from which 8.911 (the value of the *joint*

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If the lives are increased to 3, 4, 5, or any other number, the value of an Annuity on the *longest* of them is still found from the values of the *single* and *joint* lives, but in a different manner.—Those cases, however, as I have already said, being so complicated as not to admit of being explained in a few words, I shall beg leave to refer the reader for further information on the subject to the second Problem and its Corollaries in *Simpson's Doctrine of Annuities*.

S E C T. II.

On the Method of calculating Tables of the Values of Annuities on Single and Joint Lives.

WAS it *certain* that a person of a given age would live to the end of a year, the value of an Annuity of 1 *l.* on such a life would be the present sum that would increase in a year to the value of a life one year older, together with the value of the single payment of 1 *l.* to be made at the

lives of 40 and 50 by Table 6th in the Appendix) being subtracted, we have 15.629 for the number of years purchase required.

end

end of a year; that is, it would be 1 *l.* together with the value of a life aged one year older than the given life, multiplied by the value of 1 *l.* payable at the end of a year.—Call the value of a life one year older than the given life N , and the value of 1 *l.* payable at the end of a year $\frac{1}{r}$; then will the value of an Annuity on the given life on the supposition of a *certainty* be $\frac{1}{r} + \frac{1}{r} \times N = \frac{1}{r} \times 1 + N$.—But the fact is, that it is *uncertain* whether the given life will exist to the end of the year or not, this last value, therefore, must be diminished in the proportion of this uncertainty; that is, it must be multiplied by the probability that the given life will survive one year, or supposing $\frac{b}{a}$ to express this probability, it will be $\frac{b}{a} \times \frac{1}{r} \times 1 + N$.

This Theorem has been otherwise demonstrated by Mr. *Simpson* in his book on Life-Annuities, Cor. 7th, Prob. 1st, and by Dr. *Price* in his Treatise on Reversionary Payments, Note O of the Appendix *.—Its great utility will appear from the following examples.—Suppose the probabilities of life

* See also an *algebraical* demonstration of this Theorem in note A. of the Appendix to this work.

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as they are given by Dr. Price from the Bills of Mortality at Northampton *, and the rate of interest 4 *per cent.* or $r = 1.04$. By reasoning as in page 43, the value of a life aged 91 will be expressed by the single fraction $\frac{1}{2} \times .9615384 = .4807692$ †. The value of a life one year younger by this theorem will be $\frac{2}{4 \times 1.04} \times 1.4807692 = .7119$. The value of a life two years younger by the same Theorem will be $\frac{4}{6 \times 1.04} \times 1.7119 = 1.09737$. The value of a life three years younger will be $\frac{6}{8 \times 1.04} \times 2.09737 = 1.5125$, and if we proceed in this manner, the value of every younger life may be deduced from that next preceding; nor will the number of multiplications necessary to determine the values of all the lives much exceed the number of those which must otherwise be used for the single value of the youngest life.

In computing a table of the values of Annuities according to this or any other theorem it cannot but be particularly agreeable to have the truth of our operations continually proved as we go on.—I do not know that any rule has been delivered for

* See Table 3d, Appendix.

† See Table 1st, Appendix.

this purpose, on which account I shall beg leave to offer the following.—“ Find the
 “ value of 1 *l.* payable if the youngest life
 “ should live to the age of the oldest in the
 “ table. Find next the value of 1 *l.* payable if the youngest life should live to an
 “ age one year less than the oldest in the
 “ table. Multiply this last value by the
 “ value of an Annuity of 1 *l.* on the oldest
 “ life but one in the table, found in the
 “ manner just directed, and if the product
 “ is equal to the value of 1 *l.* payable on
 “ the first contingency mentioned in this
 “ rule, a demonstration arises that the value of the life is right. Again, find the
 “ value of 1 *l.* payable if the youngest life
 “ should live to an age two years less than
 “ the oldest life in the table. Multiply
 “ this by the value of an Annuity of 1 *l.*
 “ on a life likewise two years younger than
 “ the oldest life, found from the value just
 “ calculated of a life one year younger, and
 “ if the product is equal to the sum of the
 “ two values of 1 *l.* payable if the youngest
 “ life should continue to the age of the
 “ oldest, and to the age of the life one
 “ year less than the oldest, the value of this
 “ second life will be also right. In the
 “ same manner let the three values of 1 *l.*
 “ pay-

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“ payable if the youngest life should con-
 “ tinue to the several ages of the oldest,
 “ the second, and third lives, be added toge-
 “ ther, and if they are equal to the product
 “ arising from the multiplication of the va-
 “ lue of an Annuity on the life three years
 “ younger than the oldest (found from the
 “ value last calculated of a life two years
 “ younger) into the value of 1 *l.* payable
 “ if the youngest life should continue to an
 “ age also three years less than the oldest,
 “ a demonstration will be given that the va-
 “ lue of this third life has been rightly
 “ found.—Universally; if the several va-
 “ lues of 1 *l.* payable if the youngest life in
 “ the table should live to the age of the
 “ oldest, second, third, fourth, &c. lives be
 “ added together, and are found equal re-
 “ spectively to the products arising from
 “ the value of an Annuity on a life one,
 “ two, three, four, &c. years younger than
 “ the oldest life multiplied into the value
 “ of 1 *l.* payable if the youngest life should
 “ continue to an age one, two, three, four,
 “ &c. years less than the oldest life; a proof
 “ will arise of the exactness of the opera-
 “ tions as far as they have proceeded; and
 “ when we have gone in this manner
 “ through all the ages in the table, and the
 “ sum

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“ sum of the several values of 1 *l.* payable
 “ if the youngest life should continue to the
 “ ages of all the older lives is found *equal*
 “ to the value, found last of all, of the
 “ youngest life, we have a conclusive proof
 “ that the values of lives at all ages in the
 “ table have been computed without com-
 “ mitting any mistake, and are exact at least
 “ as far, or to as many places of decimals,
 “ as the equality reaches.”

This rule may perhaps be better understood from the following examples.—Let the probabilities of life be as in the Northampton Table *, and the rate of interest 4 per cent.—By reasoning as in page 44 †, the value of 1 *l.* payable if a life aged one year lives to the age of 92 is expressed by the fraction $\frac{1}{849}$ multiplied into 1 *l.* discounted for 91 years ‡, or .028182, which is equal to .0000332. Again, the value of 1 *l.* payable if the same life continues to the age of 91 is $\frac{2}{849}$ multiplied into 1 *l.* discounted for 90 years, or .029309, which is equal to .000069. The value of a Life

* See Table 3d in the Appendix.

† See also the note under Problem 4th, chap. 3d.

‡ See the value of 1 *l.* discounted for any number of years in Table 1st, Appendix.

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Annuity of 1 *l.* on a person aged 91 has been found in page 58 to be equal to .4807692, and this being multiplied into .000069 gives .0000332, which has likewise been just found to be the value of 1 *l.* payable if a life aged one year lives to 92, and, therefore, the value of the Life Annuity is proved to be true.—The value of 1 *l.* payable if a life of one year lives to be 90 is $\frac{4}{849}$ multiplied into 1 *l.* discounted for 89 years, (or .030481,) which is equal to .0001436. The value of a life aged 90 calculated from the value of a life aged 91 in the manner directed in page 58, is .7119. These multiplied into each other will be equal to .0001022, which is also the sum of the values of 1 *l.* payable if a life aged one year continues to the several ages of 91 and 92; consequently the value of this Life-Annuity is also true.—Again, the sum of the three values of 1 *l.* payable if a life of one year continues to the several ages of 90, 91, and 92 is .0002458. The value of a life aged 89 calculated from the value of a life aged 90 is 1.09737; and the value of 1 *l.* payable if a life aged one year continues to the age of 89 is $\frac{6}{849}$ multiplied into 1 *l.* discounted for 88 years (or into .0317) which
is

is equal to .000224. And 1.09737 multiplied into .000224 is equal to .0002458, or the sum of the values of 1 *l.* just found.—From hence it follows that 1.09737 is the true value of an Annuity of 1 *l.* on a life aged 89 years.—By proceeding in this manner through every age, it will appear at last that the sum of all the values of 1 *l.* payable if a life aged one year continues to the several ages of 92, 91, 90, 89, 88,—and two years, is 13.719, which corresponds *exactly*, as may be seen in the following table, with the value of an Annuity of 1 *l.* on the same life found, in consequence of computing upwards from the oldest to the youngest life, according to the theorem in page 57, and therefore the values of all the lives are strictly true*.

In making these calculations it will be convenient to dispose the operations into the following order.

* See the demonstration of this rule in note B. of the Appendix.

FIRST

64 OF ANNUITIES ON LIVES.

FIRST TABLE of the Values of Single Lives.

Age.	Values of Lives.	Values of 1l. payable, &c.	Sums.
92	0000000	.0000332	0000000
91	.480769	.0000690	.0000332
90	.711908	.0001436	.0001022
89	1.097377	.0002240	.0002458
88	1.512531	.0003106	.0004698
87	1.93271	.0004038	.0007804
86	2.16983	.0005460	.0011842
85	2.33075	.0007425	.0017302
84	2.59261	.0009539	.0024727
83	2.79011	.0012283	.0034266
82	2.96102	.0015724	.0046549
81	3.12506	.0019929	.0062273
80	3.36282	.0024446	.0082202
79	3.64096	.0029293	.0106648
78	3.94186	.0034488	.0135941
77	4.25533	.0040052	.0170429
76	4.51419	.0046626	.0210481
75	4.79106	.0053666	.0257107
74	5.07880	.0061192	.0310773
73	5.37267	.0069235	.0371965
72	5.66943	.0077823	.0441200
71	5.96681	.0086987	.0519023
70	6.26315	.0096760	.0606010
69	6.55730	.0107175	.0702770
68	6.84841	.0118269	.0809945
67	7.13585	.0130079	.0928214
66	7.41916	.0142645	.1058293
65	7.69802	.0156007	.1200938
64	7.97220	.0170211	.1356945
63	8.24154	.0185300	.1527156
62	8.50594	.0201326	.1712456
61	8.76533	.0218336	.1913782

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Age.	Values of Lives.	Values of 1 l. payable, &c.	Sums.
60	9.01970	.0236385	.2132118
59	9.26904	.0255529	.2368503
58	9.51338	.0275826	.2624032
57	9.75275	.0297338	.2899858
56	9.98721	.0320130	.3197196
55	10.21681	.0344269	.3517326
54	10.44163	.0369827	.3861595
53	10.66175	.0396879	.4231422
52	10.87724	.0425503	.4628301
51	11.08819	.0455782	.5053804
50	11.25491	.0489527	.5509586
49	11.42161	.0525242	.5999113
48	11.62641	.0561166	.6524355
47	11.82645	.0599124	.7085521
46	12.02188	.0639221	.7684645
45	12.21282	.0681567	.8323866
44	12.39942	.0726278	.9005433
43	12.58179	.0773475	.9731711
42	12.76005	.0823286	1.0505186
41	12.93432	.0875844	1.1328472
40	13.10472	.0931290	1.2204316
39	13.30703	.0987116	1.3135606
38	13.50266	.1045918	1.4122722
37	13.69198	.1107846	1.5168640
36	13.87528	.1173053	1.6276486
35	14.05285	.1241705	1.7449539
34	14.22945	.1313973	1.8691244
33	14.39185	.1390035	2.0005217
32	14.55378	.1470079	2.1395252
31	14.71095	.1554303	2.2865331
30	14.86358	.1642913	2.4419634
29	14.97797	.1740040	2.6062547
28	15.09091	.1842335	2.7802587
27	15.20236	.1950015	2.9644922
26	15.31231	.2063363	3.1594937
25	15.42074	.2182658	3.3658300

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Age.	Values of Lives.	Values of 1 l. payable, &c.	Sums.
24	15.52765	.2308196	3.5840958
23	15.63303	.2440284	3.8149154
22	15.73689	.2579246	4.0589438
21	15.83922	.2725436	4.3168684
20	15.94004	.2879162	4.5894120
19	16.07007	.3035028	4.8773282
18	16.22701	.3192710	5.1808310
17	16.40904	.3351862	5.5001020
16	16.58389	.3518634	5.8352882
15	16.75190	.3693391	6.1871516
14	16.91339	.3876497	6.5564907
13	17.06868	.4068341	6.9441404
12	17.21805	.4269330	7.3509745
11	17.33100	.4487846	7.7779075
10	17.44042	.4717013	8.2266921
9	17.51606	.4965939	8.6983934
8	17.53046	.5245133	9.1949873
7	17.51879	.5548026	9.7195006
6	17.42642	.5895802	10.2743032
5	17.26414	.6292728	10.8638834
4	16.99078	.6764330	11.4931562
3	16.62953	.7318042	12.1695892
2	15.77754	.8177041	12.9013934
1	13.71908	1.0000000	13.7190975

In the first of these columns is the age.—
In the second, the value of a life at that age agreeable to the *Northampton* Table of Observations *, found by calculating upwards from the oldest to the youngest in the manner already directed. That is (reckoning

* See Table 3d, Appendix.

interest at 4 per cent.) $000000 = \frac{0}{1} \times \frac{1}{1.04}$

----- $.480769 = \frac{1}{2} \times \frac{1}{1.04}$ ----- $.711908 =$

$\frac{2}{4} \times \frac{1}{1.04} \times 1 + .480769$ ----- $1.09737 =$

$\frac{4}{6} \times \frac{1}{1.04} \times 1 + .711908, \&c. \&c.$

In the third column is the value of 1 l. payable to a life aged one year (supposed the youngest life to which the operations are to be carried) provided it should exist to the ages 92, 91, 90, 89, &c. — That is; .0000332 is the probability that a life aged one year shall exist till 92 multiplied by the value of 1 l. payable at the end of 91 years; or, $.0000332 = \frac{1}{849} \times .02818$. — In

like manner; $.000069 = \frac{2}{849} \times .02931$ -----

$.0001436 = \frac{4}{849} \times .030481^*$, &c. &c.

In column 4th are the sums of all the values in the third column *above* the values even with those sums; or (which comes to the same) the sums of the two numbers immediately above them, and even with one another in column 3d and column 4th.

That is; $.0000332 = .0000332 + .0000000$

----- $.0001022 = .0000690 + .0000332$ -----

$.0002458 = .0001436 + .0001022, \&c. \&c.$

* See Table 1st, Appendix.

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In consequence of this arrangement it will be always found that the product of every number in the second column multiplied by the number even with it in the third column gives the number even with both in the fourth column; and this forms a proof during the whole progress of the calculations, that there has been no error in any of the preceding calculations; and that the value of every life, as it comes out, may be depended on to as many places of decimals, as the decimal figures in the product just mentioned are the same with the decimal figures in the correspondent number in column 4th.—For example. 1.097377 (the value of a life aged 89) multiplied by the number even with it in the next column, or by $.000224$, gives $.0002458$, which being the same to the last place of decimals with $.0002458$ in the fourth column, proves that all the preceding operations have been right, and that the values of the life aged 89 and of all the lives from the oldest to 89 may be depended on to the six places of decimals.

This proof makes a little more labour necessary in calculating the values of lives; but this addition of labour is more than compensated by the time and trouble that must

must be saved in consequence of detecting every error as it arises; and also by the satisfaction attending a series of operations, every one of which, if right, proves the truth of all that have gone before it.—The following expedient will, however, remove one half of the labour.—In order to find the numbers in the third column one multiplication is necessary and one division.—All the time employed in the latter may be saved by making the number 1000 the divisor instead of the number opposite, in the Table of Observations, to the youngest age. For example. In all the operations which bring out the numbers in column 3d, 849 is the divisor; but it might have been taken to be 1000; and then the first number in that column, instead of .0000332, would have been $.00002818 = \frac{1}{1000} \times .02818$. The second number, instead of .0000690 would have been $.00005862 = \frac{2}{1000} \times .02931$. The third instead of .0001436 would have been $.00012192 = \frac{4}{1000} \times .030481$, &c. &c.—And the correspondent sums in the fourth column would have been .0000000 ---- .00002818 ---- .00008680 ---- .00020872 ---- &c. &c.—The Table then would have been as follows.

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SECOND TABLE of the Values of Single Lives.

Age.	Values of Lives.	Values of 1 l. payable, &c.	Sums.
92	00000000	.00002818	.00000000
91	.480769	.00005862	.00002818
90	.711908	.00012192	.00008680
89	1.097377	.00019020	.0002087
88	1.512531	.00026376	.0003989
87	1.93271	.0003428	.0006626
86	2.16983	.0004635	.0010054
85	2.33075	.0006304	.0014689
84	2.59261	.0008099	.0020993
83	2.79011	.0010429	.0029092
82	2.96102	.0013349	.0039521
81	3.12506	.0016920	.0052870
80	3.36282	.0020755	.0069790
79	3.64096	.0024870	.0090545
78	3.94186	.0029281	.0115415
77	4.25533	.0034005	.0144696
76	4.51419	.0039588	.0178701
75	4.79106	.0045563	.0218289
74	5.07880	.0051953	.0263852
73	5.37267	.0058781	.0315805
72	5.66943	.0066072	.0374586
71	5.96681	.0073852	.0440658
70	6.26315	.0082149	.0514510
69	6.55730	.0090992	.0596659
68	6.84841	.0100411	.0687651
67	7.13585	.0110438	.0788062
66	7.41916	.0121106	.0898500
65	7.69802	.0132451	.1019606
64	7.97220	.0144509	.1152057
63	8.24154	.0157321	.1296566
62	8.50594	.0170926	.1453887

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Age.	Values of Lives.	Values of 1 l. payable, &c.	Sums.
61	8.76533	.0185368	.1624813
60	9.01970	.0200692	.1810181
59	9.26904	.0216945	.2010873
58	9.51338	.0234177	.2227818
57	9.75275	.0252440	.2461995
56	9.98721	.0271791	.2714435
55	10.21681	.0292285	.2986226
54	10.44163	.0313983	.3278511
53	10.66175	.0336951	.3592494
52	10.87724	.0361253	.3929445
51	11.08819	.0386960	.4290698
50	11.25491	.0415609	.4677658
49	11.42161	.0445931	.5093267
48	11.62641	.0476430	.5539198
47	11.82645	.0508657	.6015628
46	12.02188	.0542699	.6524285
45	12.21282	.0578650	.7066984
44	12.39942	.0616610	.7645634
43	12.58179	.0656681	.8262244
42	12.76005	.0698970	.8918925
41	12.93432	.0743591	.9617895
40	13.10472	.0790665	1.0361486
39	13.30703	.0838062	1.1152151
38	13.50266	.0887985	1.1990213
37	13.69198	.0940560	1.2878198
36	13.87528	.0995923	1.3818758
35	14.05285	.1054208	1.4814681
34	14.22494	.1115563	1.5868889
33	14.39185	.1180139	1.6984452
32	14.55378	.1248097	1.8164591
31	14.71095	.1319604	1.9412688
30	14.86358	.1394834	2.0732292
29	14.97797	.1477305	2.2127126
28	15.09091	.1564143	2.3604431
27	15.20236	.1655564	2.5168574

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Age.	Values of Lives.	Values of 1 l. payable, &c.	Sums.
26	15.31231	.1751795	2.6824138
25	15.42074	.1853077	2.8575933
24	15.52765	.1959658	3.0429010
23	15.63303	.2071801	3.2388668
22	15.73689	.2189780	3.4460469
21	15.83922	.2313882	3.6650249
20	15.94004	.2444409	3.8964151
19	16.07007	.2576738	4.1408560
18	16.22701	.2710611	4.3985298
17	16.40904	.2845731	4.6695909
16	16.58389	.2987323	4.9541640
15	16.75190	.3135689	5.2528963
14	16.91339	.3291146	5.5664652
13	17.06868	.3454022	5.8955798
12	17.21805	.3624662	6.2409820
11	17.33100	.3810182	6.6034482
10	17.44042	.4004744	6.9844664
9	17.51606	.4216082	7.3849408
8	17.53046	.4453118	7.8065490
7	17.51879	.4710274	8.2518608
6	17.42642	.5005436	8.7228882
5	17.26414	.5342526	9.2234318
4	16.99078	.5742916	9.7576844
3	16.62953	.6213018	10.3319760
2	15.77754	.6942307	10.9532778
1	11.64748	1.0000000	11.6475085

In this way the same proof of all the operations is obtained; nor has any error arisen except in the value of the life last calculated, which is very easily corrected by taking the number of the living at the age of this life agreeable to the Table of Observations,

vations, instead of supposing it 1000: that is, by making the value of the life aged one year $\frac{722}{849} \times \frac{1}{1.04} \times 1 + 15.77754$ instead of $\frac{722}{1000} \times \frac{1}{1.04} \times 1 + 15.77754$.

The use of logarithms would further expedite the calculations; and by these means a skilful person may, in three or four hours, calculate the values of single lives at all ages, agreeably to any given table of observations, without the possibility of committing any mistakes*.

Having said so much on the values of Annuities on *Single Lives*, the method of finding the values of Annuities on the *joint continuance* of any single lives will be easily understood.—Call the value of any two joint lives M, the probability that two lives one year younger will exist a year $\frac{bd}{ac}$ and $\frac{1}{r}$, as before, the value of 1 l. payable at the end of the year. Then by reasoning as in page 57, the value of the joint continuance of two lives one year younger will be expressed by $\frac{bd}{ac} \times \frac{1}{1 + M}$.—This, together with the necessary proof of the operations,

* The two following Tables have been computed in this manner,

will

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will be sufficiently explained by the following specimens.

SPECIMEN of the easiest Method of computing the Value of any two Joint Lives.

Let the two lives be equal; the rate of interest 4 per cent. or $r = 1.04$; and the table of observations the *Northampton Table*, given in the Appendix.

Ages.		Values of two equal joint lives.	Values of 1 l. payable, &c.	Sums.
92	92	.000000	.00000002818	.0000000000
91	91	.240383	.00000011723	.00000002818
90	90	.298169	.00000048768	.00000014541
89	89	.55477	.00000114120	.00000063309
88	88	.84092	.00000211008	.00000177429
87	87	1.13287	.0000034280	.00000388437
86	86	1.21351	.0000060255	.00000731237
85	85	1.24462	.0000107168	.00001333787
84	84	1.4143	.0000170079	.00002405467
83	83	1.5144	.0000271154	.0000410625
82	82	1.5998	.0000427168	.0000681779
81	81	1.6868	.0000659880	.0001108947
80	80	1.8570	.0000954730	.0001768827
79	79	2.0694	.0001318110	.0002723557
78	78	2.2976	.0001756860	.0004041667
77	77	2.5427	.0002278335	.0005798527
76	76	2.7182	.0002970100	.0008076862
75	75	2.9192	.0003781629	.0011046962
74	74	3.1364	.0004727723	.0014828591
73	73	3.3605	.0005819219	.0019556314
72	72	3.5893	.0007069704	.0025375533

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Ages		Values of two equal joint lives.	Values of 1 l. payable, &c.	Sums.
71	71	3.8203	.0008492980	.0032445237
70	70	4.0516	.0010104327	.0040938217
69	69	4.2821	.0011919952	.0051042544
68	68	4.5111	.0013957129	.0062962496
67	67	4.7381	.0016234386	.0076919625
66	66	4.9626	.001877143	.0093154011
65	65	5.1843	.002158951	.011192544
64	64	5.4031	.002472104	.013351495
63	63	5.6188	.002816055	.015823599
62	62	5.8313	.003196316	.018639654
61	61	6.0407	.003614776	.021835970
60	60	6.2468	.004074048	.025450746
59	59	6.4497	.004577540	.029524794
58	58	6.6494	.005128476	.034102334
57	57	6.8459	.005730388	.039230810
56	56	7.0392	.006387088	.044961198
55	55	7.2294	.007102526	.051348286
54	54	7.4165	.007880973	.058450812
53	53	7.6006	.008727031	.066331785
52	52	7.7818	.009645455	.075058816
51	51	7.9599	.01064140	.084704271
50	50	8.0779	.01180330	.09534567
49	49	8.2008	.01306578	.10714897
48	48	8.3829	.01434054	.12021475
47	47	8.5608	.01571750	.13455529
46	46	8.7350	.01720355	.15027279
45	45	8.9054	.01880613	.16747634
44	44	9.0723	.02053311	.18628247
43	43	9.2358	.02239282	.20681558
42	42	9.3960	.02439405	.22920840
41	41	9.5532	.02654020	.25360245
40	40	9.7065	.02885927	.28014865
39	39	9.9110	.03117591	.30900792
38	38	10.1073	.03365463	.34018383
37	37	10.2962	.03630562	.37383849

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Ages.		Values of two equal joint lives.	Values of 1 l. payable, &c.	Sums.
36	36	10.4782	.03913977	.41014411
35	35	10.6538	.04216832	.44928388
34	34	10.8240	.04540341	.49145220
33	33	10.9880	.04885776	.53685561
32	32	11.1462	.05254488	.58571337
31	31	11.3007	.05647905	.63825825
30	30	11.4500	.06067528	.69473730
29	29	11.5422	.06544461	.75541258
28	28	11.6351	.07054285	.82085719
27	27	11.7292	.07599039	.89140004
26	26	11.8242	.08180883	.96739043
25	25	11.9191	.08802116	1.04919926
24	24	12.0139	.09465148	1.13722042
23	23	12.1089	.10107254	1.23187190
22	22	12.2038	.1092700	1.3335973
21	21	12.2985	.1173138	1.4428673
20	20	12.3928	.1258871	1.5601811
19	19	12.5346	.1345057	1.6860682
18	18	12.7201	.1431203	1.8205739
17	17	12.9461	.1516775	1.9636942
16	16	13.1616	.1607180	2.1153717
15	15	13.3673	.1702679	2.2760897
14	14	13.5638	.1803548	2.4463576
13	13	13.7516	.1910074	2.6267124
12	12	13.9311	.2022562	2.8177198
11	11	14.0530	.2148943	3.0199760
10	10	14.1709	.2282704	3.2348703
9	9	14.2356	.2432679	3.4631407
8	8	14.2031	.2609527	3.7064086
7	7	14.1320	.2807323	3.9673613
6	6	13.9355	.3048311	4.2480936
5	5	13.6352	.3339079	4.5529247
4	4	13.1723	.3709924	4.8868326
3	3	12.5931	.4175148	5.2578250
2	2	11.3227	.5012346	5.6753398
1	1	6.1765*	1.0000000	6.1765744

* The true value of these two joint lives is 10.5432.

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In the third column the values are $\frac{1}{2} \times \frac{1}{1.04} = .00060000$ ----- $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{1.04}^* \times \frac{1}{1.04} = .240383$ ----- $\frac{2}{4} \times \frac{2}{4} \times \frac{1}{1.04} \times \frac{1}{1.04} = .298169$ ----- $\frac{4}{6} \times \frac{4}{6} \times \frac{1}{1.04} \times \frac{1}{1.04} \times 1 + .298169 = .554772$ ----- $\frac{6}{8} \times \frac{6}{8} \times \frac{1}{1.04} \times \frac{1}{1.04} \times 1 + .554772 = .84092$ --- and so on.

In the fourth column the first value is the probability that two lives aged one year will *both* exist to the age of 92 multiplied by the value of 1 *l.* payable at the end of 91 years; or (taking 1000 for the number living at one year of age) $\frac{1}{1000} \times \frac{1}{1000} \times .028182 \dagger = .000000028182$. ----- The second value is $\frac{2}{1000} \times \frac{2}{1000} \times .029309 = .00000011723$; or $\frac{2}{1000}$ multiplied by .00005862 (the second number in the third column of the second Table of the values of *single* lives, page 70.) Also the third value is $\frac{4}{1000}$ multiplied by .00012192, the

* Had $\frac{1}{2} \times \frac{1}{1.04}$, $\frac{2}{4} \times \frac{1}{1.04}$, $\frac{4}{6} \times \frac{1}{1.04}$, &c. been preserved from the computation of single lives in page 70, a further saving of labour, perhaps, might have been produced.

† See Table 1st, Appendix.

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third number in the third column of the same Table. The fourth value is $\frac{6}{1000}$ multiplied by .0001902, the fourth number in the third column of the same table; and so on.

In the fifth column are the sums respectively of .00000002818 and 0000000; of .00000011723 and .00000002818, &c. —And the equality of the products of the values even with one another in columns third and fourth to the numbers in the same line in the fifth column forms the same proof of the accuracy of the operations with that already explained in giving an account of the method of computing the values of Single Lives.

SECOND SPECIMEN.

Let the two lives be unequal, the difference of age 60 years, and the rate of interest and table of observations the same.

Ages.		Values of two joint Lives.	Values of 1 l. payable, &c.	Sums.
92	32	.0000000	.00012481	00000000
91	31	.472906	.00026392	.00012481
90	30	.69673	.00055793	.00038873
89	29	1.0680	.00088638	.00094666
88	28	1.4649	.00125131	.00183304
87	27	1.8630	.00165556	.00308435
86	26	2.0813	.00227733	.00473991

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Ages.	Values of two joint Lives.	Values of 1 l. payable, &c.	Sums.	
85	25	2.2275	.00315023	.00701724
84	24	2.4706	.00411528	.01016747
83	23	2.6515	.00538668	.01428275
82	22	2.8070	.00700730	.01966943
81	21	2.9561	.00902414	.02667673
80	20	3.1750	.01124428	.03570087
79	19	3.4375	.01365671	.04694515
78	18	3.7262	.01626367	.06060186
77	17	4.0315	.01906640	.07686553
76	16	4.2818	.02240492	.09593193
75	15	4.5469	.02602622	.11833685
74	14	4.8203	.02994943	.14436307
73	13	5.0977	.03419482	.17431250
72	12	5.3762	.03878388	.20850732
71	11	5.6437	.04381709	.24729120
70	10	5.9098	.04925835	.29110829
69	9	6.1626	.05523067	.34036664
68	8	6.3911	.06189834	.39559731
67	7	6.6073	.06924103	.45749565
66	6	6.7891	.07758426	.52673668
65	5	6.9396	.08708317	.60432094
64	4	7.0405	.09820386	.69140411
63	3	7.1000	.11121302	.78960797
62	2	6.9390	.12982114	.90082099
61	1	1.0306*	1.00000000	1.03064213

Here in the third column ooooo = $\frac{179}{187} \times$

$$\frac{0}{1} \times \frac{1}{1.04} \text{ ---- } .472906 = \frac{187}{195} \times \frac{1}{2} \times \frac{1}{1.04} \times$$

$$1 + 0 \text{ ---- } .69673 = \frac{195}{203} \times \frac{2}{4} \times \frac{1}{1.04} \times$$

* The true value of these two joint lives is 6.2254.

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$$\frac{1 + .472906}{1 + .69673} \text{ --- } 1.068 = \frac{203}{211} \times \frac{4}{6} \times \frac{1}{1.04} \times$$

and so on.

$$\begin{aligned} \text{In the fourth column } .00012481 &= .12481 \\ \times \frac{1}{1000} \text{ ---- } .00026392 &= .1319604 \times \frac{2}{1000} \\ \text{---- } .00055793 &= .1394834 \times \frac{4}{1000}, \text{ and} \\ &\text{so on.} \end{aligned}$$

In computing this fourth column, the number of the living at each of the youngest ages to which the values are to be found (that is, at the ages 1 and 61) are supposed to be 1000; and it appears in both these examples that half the trouble of computing the numbers in this column is saved by having previously computed the Table (in page 70) of the values of single lives, and finding there the numbers .12481-- .1319604 -- .1394834 opposite to the ages 32, 31, 30, &c. in the second specimen, and the numbers .00002818, .00005862, .00012192 opposite to the ages 92, 91, 90, &c. in the first specimen. The proof in all these examples is the same.

The calculation of the values of joint lives in exact agreement to a given table of observations has been considered as a task so great that it has been seldom undertaken. —We have therefore hitherto been obliged to

to satisfy ourselves (as I have before observed, page 50) with the values of joint lives calculated by an easy rule given in the next chapter, Problem 3d, founded on Mr. *De Moivre's* hypothesis of an equal decrement of human life through all its stages; but this hypothesis not corresponding sufficiently to fact, the values of joint lives derived from it are not sufficiently correct; and when combined (as they must be in computing the values of Reversions and of Annuities dependent on Survivorships) with other inaccuracies derived from the same hypothesis, produce errors in the solution of many questions which are too considerable.—The excellent Mr. *Simpson* has indeed given a Table of the values of two joint lives agreeable to the London Table of Observations; but this table of observations ought not to be much used, because, representing the rate of mortality among the inhabitants of London taken in the gross, it gives the values of lives much too low for the middling and the better sort of people in London itself.—It is necessary I should add, that Mr. *Simpson's* Tables of the values of lives in London give them only to one place of decimals, whereas it is in many cases absolutely necessary that they should

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be given to at least two or three places of decimals.—I have, therefore, in the preceding pages been more minute than I should otherwise have been in describing the safest and most expeditious methods of calculating such values.

And, as by these methods the calculations are rendered pleasant as well as expeditious, I hope that ere long some person will undertake them, chusing for his guide the *Northampton* Table of Observations, which perhaps is better fitted for common use than any other.

CHAP.

C H A P. III.

An Account of the Method of Computing in all Cases the Values of Assurances, and of Reversions depending on any one, two, or three Lives, or any Survivorships between them.

INTRODUCTION.

IN the two preceding chapters I have endeavoured to explain the principles from which the values of Assurances and Annuities are deduced, without entering into the solution of any cases in particular, but referring occasionally to the writings of those mathematicians where they may be found in their greatest accuracy.—The Problems in this chapter are connected no further than any other Problems with the observations in the preceding chapters; all of them being determined from the same common principles.

The eighteenth of the following Problems is the same with the 38th in Mr. *Simpson's* Select Exercises; and from the solution of this Problem, the answers to the 19th, 20th, 21st, 22d, 24th, 25th, and 26th are partly derived.

The 23d, 27th, 28th, and all the subsequent Problems for determining the values of Reversions depending on Survivorships between three lives for their whole duration and for terms, are no ways deduced from the solution of Mr. *Simpson's* Problem. These, together with the Problems just mentioned, comprehend, as far as I can perceive, all the different cases of questions in which any Survivorship between three lives can be involved.

The 4th, 5th, 6th, 7th, 8th, 9th, 14th, and 15th Problems (the 4th, 8th, and 9th taken from Dr. *Price*, and the rest from Mr. *Simpson*) have been chiefly inserted as necessary to the solution of those that follow: And, in order to render the work more complete, I have added *Corollaries* to some of these Problems of Dr. *Price* and Mr. *Simpson*, from the solutions of which and their respective Problems, the values of most Assurances may be determined wherein one or two lives are concerned.

The

The 10th, 12th, 14th, and 15th Problems contain the solutions of cases which have lately offered themselves in practice, and, therefore, are not improperly introduced here.—In short, the greatest part of this chapter is new. Most of the Problems have never been solved before, especially those which determine the values of Reversions, depending on Survivorships between three lives, *for terms*.—Mr. *Dodson* in his *Mathematical Repository* has, indeed, given the solutions of some of them for the *whole duration of life* and in *one present* payment; but these solutions are rendered in a great measure useless, not only as they are given in *algebraical symbols*, which discourage the generality of readers who are not acquainted with the mathematics, but also as they are derived from an hypothesis which produces very considerable errors when more than two lives are involved in the survivorship. I have therefore, endeavoured to give those solutions, so as to be adapted to any rate of mortality; and, to the Rules for determining the values in *single* payments, I have also added other Rules for determining the same values in *annual* payments.

With regard to the demonstrations of the following Problems; it is evident, that all

those which derive their solutions from Dr. *Price's* and Mr. *Simpson's* Problems require no explanation.—Such, however, as are not connected with any of the fore-mentioned Problems, and whose solutions are consequently not so obvious, I have demonstrated in the *Appendix*; thinking it best, in order to avoid deterring readers who are not mathematicians, to exclude all *analytical operations* from the body of the work.

The Tables in the Appendix have been inserted merely for the sake of illustrating the Problems by examples.—On this account they are confined to one rate of interest; but if the calculations are required at a higher or lower rate; other Tables will be found in most books on the subject; and the application of them being understood in one case, will easily be understood in every other.

In the foregoing chapter I have given an account of the most expeditious and easy method of computing tables of the values of single and joint lives in exact agreement to the decrements of life or rate of mortality in any particular place. The labour of computing such tables has been hitherto reckoned so great, that (except by Mr. *Simpson* for *London*) it has never been undertaken.

taken.—Having then no such tables, except for the very worst lives and carried only to one place of decimals, I have been the more desirous, in order a little to supply this defect, to compute those in the preceding section.—But these computations having been meant chiefly as *specimens*, and confined to one rate of interest and to the *Northampton* Table of Observations, it becomes necessary, in order to prepare the way for what is to follow in this chapter, to explain an easy method of *approximating* to the value of any life or lives according to any given rate of mortality. With this view I shall introduce the three first Problems, whose solutions afford the best and the only means, I am acquainted with, for supplying the want of the *real* values with any tolerable exactness.

PROBLEM I.

To find the *expectation* of any given life according to any given Table of the decrements of life.

SOLUTION.

Find the sum of all the living in the Table from the age of the given life. Divide this sum by the number of the living at that age; and the quotient lessened by half unity will be the required expectation.

EXAMPLE.

The sum of all the living from the age of ten in the *Northampton* Table of Observations (see Appendix, Table 3d) is 22811. This sum divided by 570, the number living at the age of ten, gives 40.02, which lessened by *half-unity* gives 39.52, or the expectation of a life of the age of ten according to the *Northampton* Table of Observations.

The *expectation* of a life is the same with its value supposing money not to bear interest; and in *De Moivre's hypothesis*, it is always *half the difference* between the age of a life and 86.

The *expectation* of a life at a given age, is also the same with the *time* or *number of years* which one with another a body of people at that age will live according to a given

given table of observations, some living as much *longer* as others live *less* than that time,

PROBLEM II.

To approximate to the value of a given life according to any given table of observations,

SOLUTION.

Find in Table 2d the value of an Annuity certain for a number of years equal to the *complement*, that is, to double the *expectation* of the life. Multiply this value by the perpetuity increased by unity, and divide the product by the complement.—The *quotient* subtracted from the perpetuity will be the value.

EXAMPLE.

The expectation of a life aged ten, by Table 4th, is 39.52. Double this expectation, or the *complement* of the life, is 79.04. —The value of an Annuity certain for 79.04 years is (by Table 2d) 23.874. The product of 23.874 into 26 (or the perpetuity increased by unity) is 620.724; which
divided

divided by 79.04 (the complement) gives 7.85. And this quotient subtracted from 25 (the perpetuity) gives 17.15 years purchase, the value *nearly* of a life aged ten according to the *Northampton* Table of Observations.—The same value as calculated in exact agreement to the *Northampton* Table is (by Table, page 64) 17.44.

The value found by the same rule of a life aged ten according to the *London* Table of Observations * is 16.27 years purchase; and this value as given by Mr. *Simpson* in exact agreement to the *London* Table is 16.4.

The value found by the rule of a life aged twenty, according to the *Northampton* Table of Observations, is 15.93 years purchase; and of a life aged thirty, 14.79.—These values calculated in *exact* agreement to the *Northampton* Table are (by the Table, page 64) 15.94 and 14.86.

SCHOLIUM.

This rule gives the values of lives on the hypothesis of an equal decrease in the probabilities of living; and it is by a rule,

* See *Simpson's* Select Exercises, Table 2d and 5th, page 255 and page 260.

which

which agrees with this, that Mr. *De Moivre* has calculated the table commonly used of the values of lives according to his *hypothesis*. —By this table the approximation required by the foregoing Problem may be found more compendiously in the following manner. “ Take the difference between the
“ complement of the life and 86; and the
“ value deduced from Mr. *De Moivre*’s
“ Table corresponding to that difference,
“ provided it is not less than ten, will be
“ the value of the life.”

Thus; the complement of a life aged 15 being 73 by the *Northampton Table* *, the difference between it and 86 is 13, and the value corresponding to 13 in Mr. *De Moivre*’s *Table* of the values of lives †, is 16.60 reckoning interest at 4 *per cent.*; and this is nearly the value of a life aged 15 by the *Northampton Table* ‡.

PROBLEM III.

To approximate to the value of two joint lives, according to any given table of observations.

* See Table 4th, Appendix.

† See Table 5th, Appendix.

‡ See Table, page 64.

SOLU-

SOLUTION.

Find the complements of the two lives, according to the table, by Problem 1st.—Find the *difference* between the complement of the *youngest* life and the complement of the *oldest* life increased by unity and twice the perpetuity. Multiply this difference by the value of an Annuity certain for a time equal to the complement of the *oldest* life, and by this complement divide the product, reserving the quotient.

From twice the perpetuity subtract the reserved quotient, and multiply the remainder by the perpetuity increased by unity *. This last product, divided by the complement of the *youngest* life, and then subtracted from the perpetuity, will be the required value †.

* N. B. When the complement of the youngest life is *greater* than the complement of the oldest life increased by unity and twice the perpetuity, the reserved quotient, instead of being *subtracted* from twice the perpetuity, must be *added* to it, and the *sum*, not the *difference*, multiplied by the perpetuity increased by unity.—If they are *equal*, the value of the joint lives will be “the perpetuity diminished by twice the *quotient* arising from dividing the perpetuity with its square added, by the common complement of the lives.”

† This rule is the same with that given by Dr. Price, from Mr. Simpson, in his Treatise on Reversionary Payments, page 309, 3d edition.

EXAM-

EXAMPLE.

Let the joint lives proposed be 10 and 15; and let the table of observations be Mr. *Simpson's* for *London*, and the rate of interest 4 *per cent.*—The complements of the two lives are 69.6 and 63.8 (see Mr. *Simpson's* Select Exercises, Table 2d, page 255.) The complement of the oldest life, increased by unity and twice the perpetuity, is 114.8, which lessened by 69.6 (the complement of the *youngest* life) leaves 45.2 for the *reserved remainder*.—This remainder multiplied by 22.95 (the value of an Annuity certain for 63.8 years by Table 2d, Appendix,) and the product divided by 63.8 (the complement of the *oldest* life) gives 16.26, the quotient to be reserved; which subtracted from double the perpetuity, and the remainder (or 33.74) multiplied by the perpetuity increased by unity (or by 26) gives 877.24, which divided by 69.6 (the complement of the *youngest* life) and the quotient subtracted from the perpetuity, we have 12.40 for the value nearly of two joint lives aged 10 and 15 in *London*.—The exact value (by Mr. *Simpson's* Table, Select Exercises, page 260) is 12.7.

The

The value found in the same way of two joint lives both 30, according to the decrements of life at Northampton, is 11.28; of two lives both 35 is 10.58. The exact values by the Table, page 74, are, 11.45 in the former case, and 10.65 in the latter. —If the ages are 30 and 40 the value of the joint lives (by this rule and the *Northampton* Table of Observations) will be 10.44. —If the ages are 46 and 56 the value will be 7.86.

What is said in the Scholium to the last Problem must be applied to this.—And it should be remembered further, that these approximations are to be used only when the decrements of life in a place are given, but no recourse can be had to any tables giving the values of lives agreeably to those decrements. This, as I have before observed, is at present almost universally the case; and the approximations now explained are, as far as I know, the only means that can be used to supply the defect.—Sometimes they deviate too much from exactness; but seldom more than in the examples I have given.

PROB-

PROBLEM IV.

To determine the value of an Annuity on a given life for any number of years.

SOLUTION.

Find the value of a life as many years older than the given life as are equal to the term for which the Annuity is proposed. Multiply this value by 1 $\frac{1}{2}$ payable at the end of this term, and also by the probability * that the life will continue so long.
Sub-

* The probability that a given life shall continue any number of years or attain to a *given age*, is the fraction, whose *numerator* is the number of the living in any table of observations opposite to the given age, and *denominator* the number opposite to the present age of the given life. Thus; 203 is the number, in Table 3d, Appendix, opposite to 60, and 435 the number opposite to 30— $\frac{203}{435}$ is, therefore, the probability that a person, whose age is 30, shall attain to 60, or live 30 years.

In the same manner the probability that any two lives shall continue in being together any number of years, or both attain to *given ages*, is the fraction, whose *numerator* is the product arising from the numbers of living, in the Table of Observations opposite to both those *given ages*, multiplied by one another, and *denominator*, the product arising from the numbers opposite to the *present ages* multiplied into each other.—Hence the probability that two
5 lives

Subtract the product from the present value of the given life, and the remainder multiplied by the Annuity will be the answer.

lives aged 20 and 25 shall *both* continue in being 10 years is (by Table 3d) equal to $\frac{435 \times 400}{515 \times 475} = \frac{174000}{244625}$ ----

Again : The probability that a life shall *fail* in any number of years, or *not* attain to a *given age*, is the fraction, expressing the chance of his *living* so long, subtracted from unity, agreeable to what has been observed in page 52 : In other words ; it is the fraction whose *denominator* is the number of the living opposite to the *present age*, and whose *numerator* is the *difference* between this *denominator* and the number of living at the *given age*.

Thus ; $\frac{203}{435}$ is the probability that a life of 30 shall attain

to 60— $\frac{232}{435}$, therefore, will be the probability that

he shall *not* attain to this age.—For the same reason $\frac{435}{515}$

and $\frac{400}{475}$ being the probabilities that two lives aged 20

and 25 shall severally *exist* 10 years ; $\frac{80}{515}$ multiplied into

$\frac{75}{475}$ ($= \frac{6000}{244625}$) will be the probability that they shall

both die in 10 years.—And this fraction again being sub-

tracted from unity will give $\frac{238625}{244625}$ for the probability

that they shall *not both* die in this time ; that is, that one of them at least shall *live* 10 years.

Further— $\frac{174000}{244625}$ is the probability that the lives of

20 and 25 shall *exist* together 10 years, which being sub-

tracted from unity, or the *numerator* subtracted from the

denominator, we shall have $\frac{70625}{244625}$ for the probability that

they shall *not both* live so long ; that is, that one of them at least shall *die* in this time.

EXAMPLE.

Let the Annuity be 10 *l.* The age of the given life 30 years, and the term proposed 15 years.—The value of a life aged 45 (or 15 years older than the given life) appears, by the Table in page 64 to be 12.2128. The value of 1 *l.* payable at the end of 15 years (by Table 1st, Appendix) is .5553; and the probability that the life will exist so long (by Table 3d) $\frac{325}{435}$. These three quantities multiplied into each other are equal to 5.0668, which being subtracted from 14.8635, (the present value of the given life by the Table in page 64) we have 9.7967, and this remainder multiplied again into 10, gives 97.967 *l.* for the value required.

COROLLARY FIRST.

In the same manner may the value of an Annuity for any given term upon two *joint* lives be determined.—Thus; supposing the age of these two persons to be 30 years, and the term proposed 15 years.—The value of two joint lives aged 45 is (by Table, page 74) 8.9054, which multiplied into .5553 (the value of 1 *l.* at the end of 15 years by

H

Table

Table 1st, Appendix) and also into $\frac{325 \times 325}{435 \times 435}$ (the probability * by Table 3d that *both* lives shall exist so long) will be equal to 2.7604, and this subtracted from 11.45 (the whole present value of the joint lives by the Table in page 74) leaves 8.6896 for their value during the term proposed. — And since the value of the *longest* of two lives is equal to the *joint* lives subtracted from the sum of the two *single* lives †; if from twice 9.7967 we subtract 8.6896, we shall have also 10.9038 for the value of the longest of two equal lives aged 30 for the term just mentioned. — This rule may be applied to any ages or terms, and is further explained in Dr. Price's Treatise on Reverfionary Payments, Schol. to Quest. 6th.

COROLLARY SECOND.

The value of an Annuity for the remainder of a term certain after any life or lives is very easily obtained from the foregoing Problem and Corollary; the rule being as follows. “ Find the value of an Annuity “ on the life or lives for the given term by “ the Problem or Corollary. Find also the

* See the preceding note.

† See the Rule in page 55.

“ value of an Annuity certain for the given
 “ term by Table 2d. Subtract the former
 “ of these two values from the latter, and
 “ the remainder will be the answer.” Thus;
 supposing the age of the given life to be 30
 and the term 15 years. The value of the
 single life for 15 years is found by the
 Problem to be 9.7967; and the value of an
 Annuity certain for this term is 11.1183.
 Therefore 1.3216 is the number of years
 purchase required.—In the same manner the
 value of an Annuity certain for the remain-
 der of 15 years after the extinction of two
joint lives aged 30 may be found by the as-
 sistance of the preceding Corollary to be
 equal to 2.4287; and for the remainder of
 the same term after the *longest* of two such
 lives to be equal to .2145.

REMARK. It should be observed that
 unless the value of the Annuity on the whole
 life is calculated to three places of deci-
 mals, as in the Table in page 64, those
 rules will be defective, especially if the term
 does not exceed ten years; in which case it
 will become necessary to compute the value
 of the Annuity in the Problem and first
 Corollary for each year separately, agreeable
 to the directions in page 43. But if the
 Life Annuity, as in the present examples,

is given in the Table to three or more places of decimals, the preceding solutions are sufficiently correct in all cases whatever.

PROBLEM V.

To find the value of an Annuity for three joint lives A, B, and C*.

SOLUTION.

Let A be the youngest and C the oldest of the three proposed lives. Take the value of the two joint lives B and C, and find the age of a single life D of the same value; then find the value of the joint lives A and D; which will be the answer.

EXAMPLE.

Let the three given ages be 20, 30, and 60.—The value of the two oldest joint lives B and C (by Table 6th) is 7.779, answer-

* A general account of the method of finding the values of *two* joint lives has been already given in page 48, &c. And Tables of these values for *London* have been given by Mr. *Simpson* in his *Select Exercises*; and for general use agreeable to Mr. *De Moivre's* hypothesis by Dr. *Price* in his *Treatise on Reversionary Payments*. This last Table I have inserted in the Appendix.

ing

ing to a single life D of 65 years (by Table 5th,) and the value of the joint lives A and D (by Table 6th) is 6.882, which will be the value required.

If the values of the joint lives and the single life had been taken as Mr. *Simpson* has given them for *London*, this value would have been 5.82.

PROBLEM VI.

To find the value of an Annuity upon the longest of three lives, A, B, and C.

SOLUTION.

Let A be the youngest and C the oldest of the three proposed lives. Find the value of the joint lives B and C, and find the age of a single life D of the same value.—Moreover, find by the rule in page 55 the value of the longest of the lives A and B, also that of the longest of the lives A and C, and likewise that of the longest of the lives A and D; then the last of these three values subtracted from the sum of the two former leaves the value sought.

EXAMPLE.

Let the three given ages be 20, 30, and 60.—The value of the oldest joint lives B and C (by Table 6th) is 7.779 answering to a single life D (in Table 5th) of 65 years. The value of the longest of the two lives A and B, by the same Tables, is 18.864, also that of the longest of the two lives A and C is 16.941, and that of the longest of the two lives A and D is 16.64; therefore 19.165 is the value required.

Had the values of the joint lives and the single life been computed from the *Northampton* Table of Observations by the rules in Problem 2d and 3d, this value would have been 19.005.

SCHOLIUM.

The *exact* value of an Annuity on the longest of three lives is found by “ subtracting the values of all the joint lives, “ combined *two and two*, from the sum of “ all the *single* lives added to the value of “ the *three joint* lives.” Thus; by the Table in page 74 the value of two joint lives aged 60 is 6.2468, which multiplied
into

into 3 becomes equal to 18.7404. By the Table in page 64 the value of a single life aged 60 is 9.0197, which multiplied also into 3 becomes equal to 27.0591. And, by Table 7th Appendix, the value of *three* joint lives aged 60 is 4.7826, which being added to 27.0591, and 18.7404 subtracted from the *sum*, we have 13.1013 for the *exact* value of the longest of three equal lives aged 60 by the *Northampton* Table of Observations. This value, by the rule in the preceding Problem and Mr. *De Moivre's* hypothesis, is 12.987.—In the same manner the value of three lives aged 70 by the rule in this Scholium and the *Northampton* Table of Observations may be found equal to 9.6126; and of three lives aged 75 equal to 7.6792.—By the rule in the Problem and Mr. *De Moivre's* hypothesis those values are 8.969 and 6.537 respectively.—From hence it appears that the *hypothesis* should by no means be used in finding those values after the age of 60. But with respect to the *approximation* in Problem 6th we must content ourselves with it till the values of *three* joint lives are computed from real observations. This, however, is a work of such immense labour that it is not likely ever to be performed; nor would it indeed be very

necessary if we had only Tables of the values of *two* joint lives computed from those observations; in which case the values of the longest of three lives might be always obtained sufficiently near the truth from the approximation just mentioned. The want of such tables is in a great measure supplied by the rules in Problem 2d and 3d; the values according to these differing so little from the real ones, that we shall seldom incur any material error in having recourse to them. Thus in the present instances, the value of the longest of three lives aged 70, taking the complements of life from the *Northampton* Table, is equal to 9.46, and of three lives aged 75 equal to 7.586; both of which are only about $\frac{1}{10}$ th of a year's purchase less than the *exact* values found by the same table.—Had the ages been younger this difference would have been still more inconsiderable.

PROBLEM VII.

To find the value of an Annuity granted upon three lives A, B, and C, on condition of its ceasing as soon as any two of them become extinct.

SOLU-

SOLUTION.

Find the value of each *pair* of joint lives, *viz.* of A and B, of A and C, and of B and C; then from the *sum* of those three values let twice the value of the three joint lives A, B, and C, be deducted, and the remainder will be the answer,

EXAMPLE.

Let the ages of A, B, and C respectively be 20, 30, and 60. By Table 6th the value of the joint lives

$$\left. \begin{array}{l} A, B \\ A, C \\ B, C \end{array} \right\} \text{ will be } \left\{ \begin{array}{l} 11.711 \\ 7.967 \\ 7.779 \end{array} \right.$$

the *sum* of which three numbers is 27.457. Moreover the value of the three joint lives A, B, and C, by Problem 5th is 6.882; therefore 13.693 is the value required.

Had the age of all the three lives been 60, the *exact* value by the *Northampton* Table of Observations would have been 9.175; and the same value by Mr. *De Moivre's* hypothesis is 8.99.

COROL-

COROLLARY.

The value of an Annuity for any given term upon the continuance of two lives out of three is determined much in the same manner; all that is necessary, being only to substitute in the preceding rule the value of the joint lives for the *given term*, instead of the values for *their whole duration*.

EXAMPLE.

Let the value of an Annuity be required upon three lives A, B, and C, aged 10, 10, and 70, for 15 years, on condition of its ceasing if two of them should fail in this time.—The value of two joint lives aged 10 for 15 years (by the Table in page 74 and Corollary to Problem 4th) is 9.578. The value of two joint lives aged 10 and 70 for 15 years, by the same Corollary, and the Table in page 78, is 5.768. The value of the three joint lives by Problem 5th and Corollary to Problem 4th, and also by the Tables in page 64, and page 78, is 5.53. Therefore the value of two out of the three given lives for 15 years is 10.054.

PROBLEM VIII.

To find the value of a given sum payable at the decease of A, *whenever* that shall happen. Or in other words; to find the value of an *Assurance* of any given sum on the *whole duration* of the life of A.

SOLUTION..

Subtract the value of the life from the perpetuity. Multiply the remainder by the product of the given sum into the interest of 100*l.* for a year: and this last product divided by 100*l.* increased by its interest for a year, will give the answer in a *single present* payment.—And this payment divided by the value of the life will give the answer in *annual* payments during the continuance of life.

EXAMPLE.

Let the age of the life be 20.—The sum 1000*l.*—By the table in page 64 the value of the life is 15.94, which subtracted from 25 (the perpetuity) leaves 9.06, and this remainder multiplied by the product of 1000*l.* into 4 or by 4000*l.* gives 36240, which being divided by 104 quotes 348.461*l.*
for

for the value in *one present* payment.—And this payment divided by 15.94 is 21.86 *l.* the same value in *annual* payments during the continuance of life.

The value of this Assurance by Mr. *De Moivre's* hypothesis is 350.346 *l.* in *one* payment, and 22.046 *l.* in *annual* payments.

COROLLARY.

The values of Assurances on any two or three *joint* lives, or on the *longest* of two or three, or two out of three lives are determined much in the same manner, only substituting the values of the *joint* lives, the *longest* of the lives, or of two out of three lives, instead of the value of the *single* life in the preceding solution.—For example: Let the value be required in *present* money, and also in *annual* payments of 1000 *l.* to be received on the failure of the *first* of three lives whose ages are 20, 30, and 60.—By Problem 5th the value of three joint lives (according to Mr. *De Moivre's* hypothesis) is 6.882, which subtracted from 25 and multiplied into 4000 gives 72472, and this product divided by 104 quotes 696.84 *l.* for the answer in *one present* payment.—And this payment divided by 6.882 gives 101.25 *l.*
for

for the same value in *annual* payments during the joint continuance of the three lives.

Again. Suppose the value of 1000*l.* is required as above to be received on the failure of the *last* of three lives whose ages are 20, 30, and 60.—By Problem 6th the value of an Annuity on the longest of three lives (according to Mr. *De Moivre's* hypothesis) is 19.165. This subtracted from 25 (the perpetuity) leaves 5.835, which multiplied by 4000 and then divided by 104 gives 224.42*l.* for the value of the Assurance in *one present* payment.—And 224.42*l.* divided by 19.165 quotes 11.709*l.* for the same value in *annual* payments during the continuance of the longest of the three lives.

Lastly. Suppose the value is required of 1000*l.* to be paid on the extinction of two lives out of three, whose ages are 20, 30, and 60.—By Problem 7th and Mr. *De Moivre's* hypothesis the value of an Annuity on the continuance of two out of the three given lives is 13.693. This subtracted from 25 (the perpetuity) leaves 11.307, which multiplied into 4000 and then divided by 104 quotes 434.884*l.* for the value in *one present* payment. And 434.884 divided by 13.693 gives 31.759 for the same value in *annual* payments during the continuance of
two

two out of the three lives ; or, which is the same thing, till the claim becomes due.

PROBLEM IX.

To find the value of a given sum payable at the decease of A, should that happen within a given term. Or in other words ; to find the value of an *Affurance* of any given sum on the life of A for any given term.

SOLUTION.

From the value of an Annuity certain for the given term, found by Table 2d, subtract the value of the life for that term, found by Problem 4th, and *reserve* the remainder. Multiply the value of 1 *l.* due at the end of the given term (found by Table 1st) by the perpetuity and also by the probability that the life of A shall fail in the given term; let the *product* be added to the *reserved* remainder and multiplied by the given sum ; then will this last product, divided by the perpetuity increased by unity, quote the value in *one present* payment.—And if this payment be divided by the value of the life for the given term, we shall have the same value in *annual* payments.

EXAM-

EXAMPLE.

Let the age of A be 30. The sum to be assured 1000 *l*. The term 15 years.—The value of the life of A for 15 years is, by Problem 4th and the *Northampton* Table, 9.7967, which subtracted from 11.1183 (the value of an Annuity certain for 15 years) leaves 1.3216 the remainder to be reserved.—The value of 1 *l*. to be received at the end of 15 years is .5553. The probability that the life of A shall fail in 15 years is (by Table 3d) $\frac{110}{435}$ *; and the perpetuity is 25. These numbers multiplied by one another, and 1.3216 added to the product, make 4.8319, which multiplied into 1000 (the given sum) and divided by 26 (the perpetuity increased by unity) gives 185.842 *l*. the value required in a *single present* payment.—And 185.842 *l*. divided by 9.7967 (the value of the life of A for the given term) quotes 18.969 *l*. the same value in *annual* payments for 15 years † subject to failure should the life fail.

COROL-

* See note, page 95.

† The payments in this rule are supposed to be made at the *end* of every year. But the general mode of making Assurances of this kind (and indeed of all others) is,
by

COROLLARY.

In a similar manner may the price of Assurance on any two or three *joint* lives, or on the *longest* of two or three, or two out of three lives, for any given term be calculated, only substituting the value of the *joint* lives, of the *longest* of the lives, or of *two out of three* lives, instead of the *single* life in the foregoing rule. This will be more clearly understood from the following examples.

EXAMPLE FIRST.

Let the value be required in one present payment, and also in annual payments of an Assurance of 1000 *l.* on two joint lives aged

by paying the first premium *immediately*, and the remaining ones at the *beginning* of every year after.—Under those circumstances, therefore, it is evident that the proper divisor will be the value of the life of A for one year *less* than the given term added to unity; hence, in the present case the value of the life of A for 14 years, by Problem 4th and the *Northampton* Table, is 9.387, which being increased by unity, and 185.842 *l.* divided by the *sum*, we have 17.891 *l.* for the *annual* payments, the first to be made immediately and the last at the beginning of the 15th year.—The same is to be understood of the examples to the Corollary of this Problem.—And universally: The divisor for determining the *annual* payments must be increased by unity, whenever it is proposed that the first payment should be made immediately.

30 for the term of 15 years ; that is, let the value be required of 1000*l.* payable if either of two lives aged 30 should fail in 15 years. —The value of the joint lives for 15 years (by Corollary 1st, Problem 4th, and the *Northampton* Table) is 8.6896, which subtracted from 11.1183 (the value of an Annuity certain for 15 years) leaves 2.4287 the remainder to be *reserved*. —The value of 1*l.* to be received at the end of 15 years is .5553. The probability that the life of one or other of two persons aged 30 shall fail in 15 years is $\frac{83600}{189225}$ * (by Table 3d,) and the perpetuity is 25. These numbers multiplied into one another, and 2.4287 added to the product make 8.562, which multiplied by 1000 (the given sum) and divided by 26 (the perpetuity increased by unity) gives 329.308*l.* the required value in *one present* payment. And 329.308*l.* divided by 8.6896 (the value of the joint lives for 15 years) gives 37.896*l.* the same value in *annual* payments for 15 years, subject to failure should the *joint* lives fail.

* See note, page 95.

EXAMPLE SECOND.

Let the value be required of an Assurance of 1000 *l.* for 15 years on the *longest* of two lives aged 30. Or in other words; let the value be required in *one present* payment and also in *annual* payments of 1000 *l.* to be received on the death of two persons aged 30, should they both die in 15 years.—The value of the *longest* of those two lives for 15 years (by Corollary 1st to Problem 4th and the Northampton Table) is 10.904, which subtracted from 11.118 (the value of an Annuity certain for 15 years) leaves .214 the remainder to be *reserved*. The value of 1 *l.* to be received at the end of 15 years is .5553. The probability that the lives of two persons aged 30 shall fail in 15 years is (by Table 3d) $\frac{110}{435}$ multiplied by $\frac{110}{435}$, or $\frac{12100}{189225}$ *; and the perpetuity 25.—These numbers multiplied by one another, and .214 added to the product, make 1.021, which multiplied by 1000 (the given sum) and divided by 26 (the perpetuity increased by unity) gives 39.269 *l.* the answer in a *single present* payment.—And 39.269 *l.* di-

* See note, page 95.

vided by 10.904 (the value of the longest of the two lives for 15 years) gives 3.601 *l.* the answer in *annual* payments for 15 years, subject to failure should *both* the lives fail.

EXAMPLE THIRD.

Suppose the value is required of 1000 *l.* payable on the extinction of two out of three lives, whose ages are 10, 10, and 70, should that happen in 15 years.—By the Corollary to Problem 7th and the *Northampton* Table of Observations, the value of an Annuity for 15 years upon two out of the three given lives, is 10.054, which subtracted from 11.118 (the value of an Annuity certain for 15 years) leaves 1.064 the remainder to be *reserved*.—The probability that two at least out of the three given lives shall die in 15 years (by Table 3d) is $\frac{427063}{1598508}$ *. This being

* The probability that two lives at least out of three shall die in any number of years is determined in the following manner.—“ Find the probability that each *pair* of lives shall fail in the given time; multiply each of those expressions into the probability that the third life shall exist so long. Find next the probability that the *three* lives shall fail in the given time. Add those four quantities together, and the *sum* will give the probability required.”—Thus: Suppose the ages of A, B, and C to be 10, 10 and 70 respectively; the

ing multiplied into .5553 (the value of 1 *l.* at the end of 15 years) and also into 25 (the perpetuity) as in the two preceding examples will produce 3.713, which added to 1.064, the *reserved* remainder, becomes equal to 4.777. And 4.777 multiplied into 1000 (the given sum) and then divided by 26 (the perpetuity increased by unity) gives 183.730 *l.* for the value in a *single present* payment.

183.73 *l.* divided by 10.054 (the value of an Annuity for 15 years on two out of the three given lives) quotes 18.274 *l.*—the same value in *annual* payments for 15 years

term 15 years, and the probabilities of life to be as they are in the *Northampton* Table of Observations, which is the third Table in the Appendix of this work.—By reasoning as in note, page 95, and note, page 13, the probability

That A, B will die and C live - - -	}	will be	{	$\frac{95}{570} \times \frac{95}{570} \times \frac{17}{123}$	}	equal to	{	$\frac{152425}{39962700}$
That A, C will die and B live - - -				$\frac{95}{570} \times \frac{106}{123} \times \frac{475}{570}$				$\frac{4781250}{39962700}$
That B, C will die and A live - - -				$\frac{95}{570} \times \frac{106}{123} \times \frac{475}{570}$				$\frac{4783250}{39962700}$
That A, B, and C will all die - - -				$\frac{95}{570} \times \frac{95}{570} \times \frac{106}{123}$				$\frac{956650}{39962700}$

the sum of which four expressions is $\frac{10676575}{39962700}$, or (dividing the numerator and denominator by 25) = $\frac{427063}{1598508}$.

sub-

subject to failure should two of the lives fail.

N. B. What is said in note, page 111, respecting the method of determining the value of the *annual* payments, is to be applied, under the same circumstances, to the three preceding examples.

PROBLEM X.

To determine the sum to which any given Annuity, forborn and improved at compound interest during the continuance of a given life, will amount at the extinction of that life *.

SOLUTION.

Divide unity by the value of 1 *l.* to be received at the end of as many years as are expressed by the *complement*, or twice the expectation, of the given life, and subtracting unity from the quotient, *reserve* the remainder.—Multiply the complement of the given life into the square of the interest of 1 *l.* for a year. Divide the *reserved* remain-

* See the demonstration of this Problem in note C. Appendix ; where Theorems are also given for determining the amount of the same Annuity, when it is laid up *half-yearly*, *quarterly*, or *momently*.

der by this product, and from the quotient subtract the perpetuity; then will this last remainder, multiplied into the given Annuity, produce the value required.

EXAMPLE.

Let the given Annuity be 50*l.* and the age of the given life 30 years.—By Table 4th the expectation of the life being 28.38, the complement will be 56.76; hence, by Table 1st, the value of 1*l.* to be received at the end of a term equal to this complement is .108.—Unity divided by this number quotes 9.2592, from which unity again being subtracted, we have 8.2592 for the remainder to be *reserved*.—The square of the interest of 1*l.* for a year (or .04 multiplied into itself) is .0016; which being multiplied into 56.76 (the complement of the given life) produces .0908; and 8.2592 (the *reserved* remainder) divided by this product quotes 90.960, from which 25 (the perpetuity) being subtracted, and the remainder multiplied by 50 (the given Annuity) we have 3298*l.* for the value required according to the *Northampton* Table of Observations.

The complement of a life at the age of 30, by the *London* Table of Observations, is

47.2;

47.2; and this would have given the amount of the same Annuity equal to 2304.2/. — The complement of the life by Mr. *De Moivre*'s hypothesis being 56, the amount of the Annuity may be found equal to 3209.8/.

PROBLEM XI.

To find the value of an Annuity during a given life, of 1/. payable at the end of the first year, 2/. at the end of the second year, 3/. at the end of the third year, and so on; the Annuity uniformly increasing every year till the extinction of the given life.

SOLUTION.

From the value of the given life subtract the value of two equal *joint* lives whose common age is the age of the given life; and multiply the remainder by the complement of the given life. The product will be the value required.

EXAMPLE.

Let the age of the given life be 40. The value of two equal joint lives aged 40 (by Table in page 74) is 9.71; which subtracted
 I 4 from

from 13.10, the value of the given single life (by Table in page 64) leaves 3.38. And this remainder multiplied by 45.72 (the complement of the given life by Table 4th) gives 154.53 *l.* the required value agreeable to the *Northampton* Table of Observations.

COROLLARY FIRST.

If the Annuity is to commence with any larger sum than 1 *l.* and in the same manner to go on increasing 1 *l.* every year; add to the preceding value, the value of the given life multiplied into the first payment lessened by unity, and the *sum* will be the answer.—Thus; if instead of 1 *l.* 2 *l.* 3 *l.* &c. the Annuities payable at the end of the first, second, third, &c. years had been 15 *l.* 16 *l.* 17 *l.* &c. 183.47 *l.* (or 13.105 the value of the given life by the Table in page 64 multiplied into 14) must have been added to 154.53 *l.* found above, which would have given 338 *l.* for the value in this case *.

COROL-

* It can hardly be necessary to observe that, if the Annuities at the end of the first, second, third, &c. years are constantly the *multiples* of any given number, the value in this case is equal to the product of such a number multiplied into the value found in the Problem. Thus if the Annuities were 10 *l.* 20 *l.* 30 *l.* &c. or 15 *l.*
30 *l.*

COROLLARY SECOND.

If the Annuity is not to commence till after an assigned time; the value of such an Annuity must be found on a life as many years older than the given life as are equal to the assigned time.—This value being multiplied into the probability that the life shall exist so long and also into 1 *l.* payable at the end of the assigned time will give the answer.

EXAMPLE.

Let the value be required of an Annuity beginning with 15 *l.* the first year, and increasing at the rate of 1 *l.* every year during the remainder of the life of a person now aged 25, after he has attained to the age of 40, and the first payment to be made when he has completed his 41st year.

The value of such an Annuity on a life of 40 has been found in the preceding Corollary equal to 338 *l.* The value of 1 *l.* payable at

30 *l.* 45 *l.* &c. at the end of the first, second, third, &c. years, the values in the Example to the Problem would have been equal to $(154.53 \times 10 =)$ 1545.3 *l.* and $(154.53 \times 15 =)$ 2317.95 *l.*

the

the end of 15 years (by Table 1st) is .555, and the probability that a life of 25 shall exist so long (by Table 3d) is $\frac{365}{473}$. These three quantities multiplied into each other give 144.14 *l.* for the value required in *one present* payment.

The value in *annual* payments, the first to be made immediately and the others at the beginning of each year, is found by dividing the value in *one* payment by the value, added to unity, of the given life for the assigned time. Thus; the value of a life aged 25 for 15 years (by the *Northampton* Table and the rule in Problem 4th) is 9.834, which being added to unity, and 144.14 *l.* divided by the *sum*, we have 13.303 *l.* for the preceding value in *annual* payments, the first to be made immediately, and the last at the end of the 15th year, or beginning of the 16th year.

COROLLARY THIRD.

If the Annuity, dependent on the given life, is to be continued only for an assigned time, beginning with 1 *l.* at the end of the first year, and increasing at the rate of 1 *l.* every year to the end of the term; the value may be found in the following manner.

ner.—“ From the value of the given life
 “ for the assigned term subtract the value
 “ for the assigned term of two equal *joint*
 “ lives, whose common age is that of the
 “ given life. Multiply the difference by
 “ the complement of the given life, and the
 “ product will be the answer.”

EXAMPLE.

Let the age of the given life be 30, and the assigned term 15 years. By Problem 4th and the Table in page 64 the value of a life of 30 for 15 years is 9.7967. By Corollary 1st to Problem 4th and the Table in page 74 the value of two equal *joint* lives aged 30 for 15 years is 8.6896. By Table 4th the complement, or twice the expectation, of the given life is 56.76. The difference between the two former values (or 1.107) multiplied into 56.76 gives 62.8331. for the value required.

COROLLARY FOURTH.

Suppose the Annuity after having increased during an assigned time is to become *stationary*; that is, to continue invariably the same for the remainder of the given life after the expiration of the assigned time.—In this case

case the rule for determining the value will be as follows. “ Find the value of a life as
 “ many years older than the given life as are
 “ included in the assigned term. Multiply
 “ this value into 1 *l.* payable at the end of
 “ the assigned time, and also into the proba-
 “ bility that the given life exists so long.
 “ Let this quantity again be multiplied into
 “ the stationary Annuity and *reserve* the
 “ product. Find next the value of the in-
 “ creasing Annuity by the rule in the fore-
 “ going Corollary. Let this value be added
 “ to the *reserved* product, and the *sum* will
 “ be the answer.”

EXAMPLE.

The value is required of an Annuity on the life of a person aged 30, of 1 *l.* payable at the end of the first year, 2 *l.* at the end of the second year, 3 *l.* at the end of the third year, and increasing at this rate every year, till at the end of 15 years it becomes an Annuity of 15 *l.* which is to continue invariably the same during the remainder of the given life.—By the Table in page 64 the value of a life aged 45 (or 15 years older than the given life) is 12.2128. The value of 1 *l.* payable at the end of 15 years (by Table 1st) is .5553, and the probability

bability that the life will exist so long (by Table 3d) is $\frac{325}{435}$ *. Those three quantities multiplied into each other are equal to 5.0668; which being multiplied again into 15 (the standing Annuity) gives 76.002 for the product to be *reserved*.—By the rule in the preceding Corollary the value of the increasing Annuity for 15 years appears to be 62.833. Therefore 138.835 *l.* is the value required.

COROLLARY FIFTH.

If the Annuity the first year is greater than 1 *l.* the value of the life for the given term, till the Annuity either ceases or becomes stationary, must be multiplied into this excess above 1 *l.*; and the product, added to the value of the increasing Annuity for the assigned term of 1 *l.* 2 *l.* 3 *l.* &c. will give the answer in this case. Thus; if instead of 1 *l.* 2 *l.* &c. in the Example to Corollary 3d, the Annuity had been 12 *l.* 13 *l.* &c. the value of the life for 15 years found by Problem 4th (or 9.7967) must have been multiplied by 11, and then added to 62.833, which would have made the answer 170.596 *l.*—In the same manner, if

* See note, page 95.

the

the increasing Annuity in the Example to Corollary 4th had been 12*l.* 13*l.* &c. and the *stationary* one 26*l.* 5.0668 in that Example must have been multiplied by 26, instead of 15, and 131.737, the product, added to 170.596 found above, which would have given 302.333*l.* for the value in that case*.

COROLLARY SIXTH.

If the Annuity, in the two last Corollaries, is not to commence till after an assigned time, we must proceed exactly in the same manner as in the second Corollary. Thus; let the value be required of an Annuity on the life of a person now aged 20, and to commence in 10 years, beginning with 1*l.* the first year, and increasing at the rate of 1*l.* every year for 15 years; at which period, or when he has attained to the age of 45, it is to continue invariably the same for the remainder of his life.—The value of such an Annuity on a life aged 30 appears (by Corollary 4th) to be 138.835. This multiplied into .675 (the value of 1*l.*

* If the Annuities 1*l.* 2*l.* &c. in the two last Corollaries are constantly the multiples of any number; see note, page 120.

at the end of 10 years by Table 1st) and also into $\frac{415}{515}$ * (the probability by Table 3d, that a life of 20 shall exist 10 years) gives 79.156*l.* for the value in *one present* payment. —And this value divided by 8.466 (the value, increased by unity, of the given life for 10 years by Problem 4th) quotes 9.35*l.* for the same value in *annual* payments, the first to be made immediately and the last at the end of the 10th year.

If the Annuity instead of becoming *stationary* at the end of 15 years after its commencement had entirely *ceased*, the value must have been found by Corollary 3d, which in the present case being equal to 62.833, will, when multiplied as above, give 35.824*l.* for the value in *one* payment; and this being divided by 8.466 quotes also 4.23*l.* for the same value in *annual* payments, the first to be made immediately, and the last at the end of the 10th year †.

* See note, page 95.

† Dr. Price in his Treatise on Reversionary Payments, page 302, 3d edition, has given another rule for solving this Problem. The rule here given agrees with it, but is much simpler and easier. — See the demonstration in note D, Appendix.

PROBLEM XII.

To determine what Annuity any given sum will purchase during the *joint* lives of two persons of given ages, and also during the life of the *survivor*; on condition, that the Annuity be reduced *one half* at the extinction of the joint lives.

SOLUTION.

Let twice the given sum be divided by the *sum* of the two *single* lives, and the quotient will give the Annuity to be paid during the *joint* lives; one *half* of which, therefore, is the Annuity to be paid during the remainder of the surviving life *.

EXAMPLE.

A and B whose ages respectively are 25 and 32, are desirous of sinking 1000 *l.* in order to receive an Annuity during their joint lives, and also another Annuity of *half* the value during the remainder of the surviving life: It is required to determine what Annuities should be granted them under those circumstances.—By Table in page 64 the value of the life of A is 15.421, and by

* See the demonstration in note E, Appendix.

the same Table the value of the life of B is 14.554. 2000 *l.* (or twice the given sum) being divided by 29.975 (the sum of the values of the two lives) gives 66.722 *l.* for the Annuity to be granted during the *joint continuance* of the lives; and its half, or 33.361 *l.*, is the Annuity to be paid during the life of the survivor.

COROLLARY.

The value of such an Annuity as is mentioned in the preceding Problem is found, by multiplying *half* the sum of the single lives into the Annuity to be paid during the *joint* lives. Thus, if A and B, aged 25 and 32, were in possession of an Annuity of 40 *l.* to be reduced as above, after the extinction of their joint lives, its present value would be equal to 599.48 *l.*; or 14.987 (the half of 29.975) multiplied into 40.

PROBLEM XIII.

To find the value of an Annuity certain for a given term after the extinction of any given life or lives.

K SOLU-

SOLUTION.

Subtract the value of the life or lives from the perpetuity, and *reserve* the remainder. Then say; as the perpetuity is to the present value of the Annuity certain, so is the said *reserved* remainder to a fourth proportional, which will be the number of years purchase required.

EXAMPLE FIRST.

Suppose A and his heirs are intitled to an Annuity certain for 14 years to commence at the death of B aged 25. What is the present value of A's interest in this Annuity?

By Table 5th the value of the life of B is 15.318, which subtracted from 25 (the perpetuity) leaves 9.682 for the remainder to be *reserved*. Therefore as 25 is to 10.563 (the value of an Annuity certain for 14 years by Table 2d) so is 9.682 (the *reserved* remainder) to 4.09, the number of years purchase required *.

EXAM-

* This Solution supposes the first payment of the Annuity to be made at the end of half a year from the extinction of the life. If the first payment is not to be made till the end of any longer term, discount the value found by this Problem for half a year *less* than that term, and the exact answer will be obtained.

Thus

EXAMPLE SECOND.

Suppose A and his heirs are entitled to an Annuity certain of 40 *l.* for 20 years to commence at the death of the longest of two lives aged 30 and 35; what is the value of A's interest in this Annuity?—By the rule in page 55, and computing from Tables 5th and 6th, the value of the longest of the two lives is 17.825; which subtracted from 25 (the perpetuity) leaves 7.175 for the remainder to be *reserved*.—As, therefore, 25 is to 13.59 (the value of an Annuity certain for 20 years by Table 2d) so is 7.175 (the *reserved* remainder) to 3.9; which multiplied by 40 (the given Annuity) produces 156 *l.* for the value required.

In a similar manner may the value of an Annuity certain after two or three *joint* lives be determined, by subtracting the joint lives from the perpetuity, and finding a fourth proportional as above.

Thus, 4.09 discounted for a year (that is, multiplied by the value of 1 *l.* to be received at the end of a year) is 3.932, which is the present value of an Annuity certain for 14 years, the first payment of which is to be made at the end of a year and a half after the failure of a life aged 25.

PROBLEM XIV.

Supposing an Annuity be enjoyed during the life of A, at whose decease B and his heirs have the nomination of a successor, who is likewise to enjoy the Annuity during his life; it is proposed to find the present value of the succeeding life.

SOLUTION.

Subtract the value of the life of A from the perpetuity, and *reserve* the remainder. Then say; as the perpetuity is to the present value of the life to be nominated at the decease of A, so is the said *reserved* remainder to a fourth proportional, which will be the answer.

EXAMPLE.

Let the age of A be 40, and the value of the life to succeed him worth 16 years purchase, (which nearly corresponds with the value of a life whose age is 19*.)—By Table 5th the value of the life of A is 13.196, which being subtracted from 25 (the perpetuity) leaves 11.804 for the remainder

* See Table 5th, Appendix, or the Table in page 64.

to be *reserved*. Therefore as 25 is to 16 so is 11.804 to 7.554, the number of years purchase required, according to Mr. *De Moivre's* hypothesis.

Had the value of the life of A been taken agreeable to the *Northampton* Table of Observations, the value of this succession would have been 7.613*.

R E M A R K.

This Problem appears to differ in no respect from the former, except that in one the succeeding Annuity is for a given *life*, in the other for a *term certain*. Consequently as the Solution in the preceding Problem applied not only to a *single* life but to the *longest* of any number of lives, and likewise to the *joint* lives, so in the present Problem the same application holds good under similar restrictions. This will be better explained by the following example.

E X A M P L E.

An Annuity is enjoyed during the *longest* of two lives aged 30 and 35, at whose ex-

* A rule for determining the value of any number of successive lives may be found in Mr. *Simpson's* Select Exercises, page 292.

tion B and his heirs have a right to nominate two other lives, which suppose to be worth 20 years purchase. It is required to determine the present value of this nomination. By the rule in page 55 and computing from Tables 5th and 6th the value of the longest of two lives aged 30 and 35 is 17.825, which subtracted from 25 (the perpetuity) leaves 7.175 for the remainder to be *reserved*. Therefore, as 25 is to 20 (the supposed value of the succeeding lives) so is 7.175 (the *reserved* remainder) to 5.74, the number of years purchase required.

The values of Annuities to one life after one, to one life after two, to two lives after one, to one life after two joint lives, and to two joint lives after one life, may be found by Problems 15th, 18th, 16th, 17th, 19th, respectively, in Mr. *Simpson's* Select Exercises.

COROLLARY.

In order to find the values of the forementioned Annuities for *terms*, all that is necessary is to substitute in Mr. *Simpson's* rules, the values of single and joint lives for the *given terms* instead of their values for their *whole duration*. Thus for example; suppose
fifteen

fifteen years of a lease to remain unexpired, and that A aged 30 is to enjoy the reversion of his life in the profits of the lease for what may happen to remain of this term after the extinction of the life of B aged also 30: It is required to determine the value of A's expectation.

By Mr. *Simpson's* 15th Problem the value of the Reversion of one life after another is obtained, by subtracting the two *joint* lives from the life in expectation. Therefore, by Problem 4th, Corollary 1st, the value of a single life aged 30 for 15 years according to the *Northampton* Table being 9.7967, and the value of two *joint* lives aged 30 for this term, by the same Corollary and Table, being 8.6896, the value of A's expectation will be 1.1071 year's purchase.—Had the value of the Reversion been computed agreeable to Mr. *De Moivre's* hypothesis, it would have been found equal to 1.189 years purchase.

PROBLEM XV.

To find the value of an Annuity for a term certain, and also for what may happen to remain of a given life or lives after the expiration of this term.

SOLUTION.

Find the value of a life or lives as many years older than the given life or lives as are equal to the term for which the Annuity certain is proposed. Multiply this value by 1 *l.* payable at the end of the given term, and also by the probability that the given life or lives will continue so long. Add the product to the value of the Annuity certain for the given term, and the *sum* will be the answer.

EXAMPLE FIRST.

Let the value be required of an Annuity certain for 15 years, and also for the remainder of a life now aged 30 after the expiration of this term.—By the Table in page 64 the value of a life aged 45 (or 15 years older than the given life) is 12.2128. The value of 1 *l.* payable at the end of 15 years is (by Table 1st) .5553; and the probability that the life will exist so long (by Table 3d) is $\frac{325}{435}$ *. These three numbers multiplied into each other produce 5.0668, which being added to 11.1183 (the value of an Annuity certain for 15 years by Table 2d) becomes equal to 16.1851, the number of years purchase required.

* See note, page 95.

EXAMPLE SECOND.

Suppose the value is required of an Annuity certain for 20 years, and also for the remainder of the *longest* of two lives now aged 10 and 70 after this term.—By the rule in page 55 and calculating according to the values of single and joint lives in pag. 64 and 78, the value of the longest of two lives aged 30 and 90 (or 20 years older than the given lives) may be found equal to 14.8787. The value of 1 *l.* payable at the end of 20 years (by Table 1st) is .4564, and the probability (by Table 3d) that two lives aged 10 and 70 shall not *both* fail in 20 years is $\frac{54045}{70110}$ *. These three numbers multiplied into each other produce 5.234, which being added to 13.590 (the value of an Annuity certain, by Table 2d, for 20 years) is equal to 18.824, the value required.

PROBLEM XVI.

B, who is of a given age will if he lives till the decease of A, whose age is also given, become possessed of an estate of a given value; to find the worth of his expectation in present money.

* See note, page 95.

SOLU-

SOLUTION.

Find the value of an Annuity on two equal *joint* lives whose common age is equal to the age of the oldest of the two proposed lives, which value subtract from the perpetuity, and take *half* the remainder; then say, as the expectation of duration of the younger of the two lives is to that of the older, so is the said half remainder to a fourth proportional; which will be the number of years purchase required when the life of B in expectation is the older of the two: But if B be the younger, then add the value so found to that of the joint lives A and B, and let the sum be subtracted from the perpetuity, and you will also have the answer in this case.

The value of the Reversion in *annual* payments is determined by dividing the *single present* payment by the value of the *joint* lives.

EXAMPLE FIRST.

Suppose the age of A to be 20 and that of B 30 years; and the given legacy 10 *l. per ann.* Then the value of two equal joint lives aged 30 (by Table 6th) being 11.182, and the perpetuity 25; the difference here will

will be 13.818, the *half* of which is 6.909. Therefore as 33 *, the expectation of A, is to 28, the expectation of B, so is 6.909 to 5.862 years purchase, which being multiplied by 10, the given Annuity, we have 58.62 for the required value of B's expectation in *one present* payment.

This is the answer by Mr. *De Moivre's* hypothesis. But if an answer is required by the *Northampton* Table of Observations, the value of the two equal joint lives will (by the Table, page 74) be 11.450. The expectation of the youngest life will (by Table

* The values of most of the following examples in this chapter have been computed from Mr. *De Moivre's* hypothesis; but it will be found as I go along that the Rules may be applied to any Table of Observations; always remembering that the *complement* of a life, according to any Table, is double its *expectation* of duration.

It should be further carefully remembered that the Rules for solving this and all the following Problems give the *exact* values according to Mr. *De Moivre's* hypothesis (except when the approximations in Problems 5th and 6th are used for finding the values of three *joint* lives, or three *single* lives †;) but that they are only *approximations* when the complements and values of lives are taken (as it is best they should) agreeable to a given Table of Observations.—These approximations, however, are so near when the Survivorship extends to the whole duration of the lives, that there is little reason to desire any greater accuracy.

† See Scholium to Problem 6th.

4th, Appendix) be 33.19, and of the oldest 28.38.—And these data will bring out the answer 57.93 *l*.

EXAMPLE SECOND.

Let the age of A be 30, that of B 20 years; and the rest as in the preceding example. Here the value of the joint lives A and B (by Table 6th) is 11.711, which being added to 5.862, found above, the sum is 17.573; and this subtracted from 25, the perpetuity, and multiplied into 10, gives 74.27 *l*. for the value in *one present* payment.—And 74.27 being divided by 11.711 (the value of the joint lives A and B) quotes 6.34 *l*. for the same value in *annual* payments, the first to be made at the *end* of the year.

PROBLEM XVII.

To find the value of a given estate at the death of B, provided that should happen *after* the death of A.

SOLUTION.

Find the value of an Annuity upon the *longest* of two equal lives whose common age is that of the older of the two lives A

and B, which value subtract from the perpetuity and take *half* the remainder. Then it will be, as the expectation of duration of the younger of the lives A and B is to that of the older, so is the said half-remainder to the number of years purchase required when B is the older of the two. But if B be the younger, then to the number of years purchase thus found add the value of an Annuity on the longest of the lives A and B, and subtract the sum from the perpetuity for the answer in this case.

The value of the Reversion in *annual* payments till the *claim is determined* is found by dividing the *single present* payment by the value of the two *joint* lives.—And the same value in *annual* payments till the claim *becomes due* is found by dividing the *single* payment by the life of B.

EXAMPLE.

Let the age of A be 30, and that of B 60 years. Let the given estate also be 40 *l. per annum*.—By the rule in page 55 and calculating from Tables 5th and 6th, Appendix, the value of the *longest* of two lives aged 60 each will be found equal to 11.683, which taken from 25, the perpetuity, leaves

13.317

13.317 for the remainder. Therefore it will be as 28, the expectation of A, is to 13, the expectation of B, so is 6.658, the half-remainder, to 3.091, the number of years purchase required, which being multiplied by 40 gives 123.64*l.* for the value in *one present* payment.

But if A had been 60 and B 30, the value required would have been 5.987 multiplied into 40, or 239.48*l.*; because the value of the longest of the lives A and B (by the rule in page 55 and Tables 5th and 6th, Appendix) is 15.922 years purchase.

Such is the solution by Mr. *De Moivre's* hypothesis.—Had the solution been required by the *Northampton* Table, it would have been necessary to compute (by Problem 3d) the value of two joint lives aged 30 and 60 from their expectations by the same Table, or from 28.38 and 13.13^{*}; which differing but little from the expectations by the hypothesis, the result would have been very nearly the same.—But had the ages of A and B been supposed 10 and 70, the value by the hypothesis (supposing A the oldest) would have been 5.92 years purchase or

* See Table 4th, Appendix.

236.8 *l.*—By the *Northampton* Table * 5.45 years purchase or 218 *l.*—Had both the ages been 10 the value would have been by the hypothesis 2.29 years purchase.—By the *Northampton* Table 2.14 years purchase.—Had the age of A been 75 and B 15, the value by the hypothesis would have been 6.87. By the *Northampton* Table 6.43 years purchase.

It appears from hence that, though the *Northampton* Table corresponds more nearly than any other to the hypothesis, yet the difference between the values given by it and by the hypothesis in the first and last stages of life is too great to be neglected.—And what is thus true of this Table and also of Dr. *Halley's* Table in the first and last stages of life, is true of the *London* Table through every part of life.

R E M A R K.

If instead of an *Annuity* the value of an equivalent *sum* had been required, the respective answers in the two preceding Problems should have been divided by 1 *l.* increased by its interest for a year, agreeable to the correction mentioned by Dr. *Price* in his *Treatise on Reversionary Payments*, Scholium to Question 6th.

* See Table 4th, Appendix, and the Tables in pag. 64, 74, and 78.

The

The same is likewise to be observed in all the following solutions where either of those Problems are concerned.

The Solutions of the 16th and 17th Problems include all the cases of Survivorship between two lives for their *whole duration*. But an expectation dependent on a Survivorship is often restricted to a *term of years* less than the whole duration of the lives.—I am unwilling to repeat too much of what has been written by others; and, therefore, I shall refer for the Rules, by which the values in such cases may be calculated, to Questions 15th and 16th in Dr. Price's Treatise on Reverfionary Payments.

PROBLEM XVIII.

B if he lives till the decease of A is to receive a given sum in case C is then extinct.—To determine the value of his expectation in present money.

SOLUTION.

CASE 1st. *If the life of B is the oldest of the three:* From the value of an Annuity on the life of B take the value of the two joint lives A and B; multiply the remainder by the given sum, and divide the
pro-

product by twice C's expectation; the quotient will be the value sought.

CASE 2d. *If the life A is the oldest of the three:* From the value of an Annuity for as many years of B's life as are expressed by the double of A's expectation of duration (found by Problem 4th) subtract the value of the two joint lives A and B; and multiply the remainder by the given sum, then divide the product by twice the expectation of C's duration as in the preceding case.

CASE 3d. *If the life C be the oldest of the three:* Then find the value of the life B, if older than A; otherwise find (by Problem 4th) the value of as many years thereof as are expressed by the double of A's expectation of duration; and from the value thus found let the value of the joint lives C and B be subtracted: Multiply the remainder by the given sum, then the product divided by twice A's expectation of duration will be the answer.

In either of those three cases if the present value in *one* payment be divided by the joint lives A and B, the quotient will be the same value in *annual* payments *, the first to be

* See the method by which the values in *annual* payments, in all the following Problems, have been deduced, in note K, Appendix.

made at the end of the year and the others to be continued till the claim is determined ; or (which is the same thing in this instance) till the money becomes payable, provided the claim is determined in favour of the purchaser.

EXAMPLE FIRST.

Let the age of B be 60, that of A 20, and that of C 30 years ; and let the proposed legacy be 1000 *l.*—This example, it is evident, belongs to Case 1st. Hence 7.967, the value of the joint lives A and B, by Table 6th, being subtracted from 9.017, the value of the single life B by Table 5th, leaves 1.05 for the remainder ; which being multiplied into 1000, the given sum, and divided by 56, twice the expectation of C by the hypothesis, quotes 18.75 *l.* for the value required in *one present* payment.—In order to find this value in *annual* payments till the claim is determined or becomes payable, 18.75 *l.* must be divided by 7.967, the value of the joint lives A and B, and the quotient, or 2.353 *l.* will be the answer.

Had the ages of A, B, and C respectively been 10, 70, and 50, the value of B's expectation in *present* money would have been by the *Northampton* Table of Observations
equal

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equal to 9.752 *l.*, and by Mr. *De Moivre's*
hypothesis equal to 11.194 *l.*

EXAMPLE SECOND.

Let the age of B be 20, that of A 60,
and that of C 30 years; and let the proposed
legacy be 1000 *l.*—According to the so-
lution of Case 2d the first thing to be deter-
mined is the expectation of A, which by the
hypothesis appears to be 13; its double,
therefore, is 26, and the value of an An-
nuity on the life of B for 26 years by Prob-
lem 4th, and computing agreeable to the
hypothesis, is equal to 13.235, from which
7.967, the value of the joint lives A and B
by Table 6th, being subtracted, the remain-
der is 5.268; which multiplied into 1000,
the given sum, and then divided by 56,
twice the expectation of C, quotes 94.071 *l.*
for the value required.—And 94.071 divided
by 7.967, as above, gives 11.807 *l.* for the
same value in *annual* payments, the first to
be made at the end of the year.

EXAMPLE THIRD.

Suppose the age of B to be 30 years, that
of A 20 and that of C 60. The legacy, as
in the other examples, 1000 *l.*—According

to Case 3d the value of the life of B (by Table 5th) is 14.684, from which 7.779 the value of the joint lives C and B (by Table 6th) being subtracted, and 6.905, the remainder, multiplied into 1000, the given sum, and then divided by 66, twice the expectation of A- by the hypothesis, quotes 104.62 *l.* for the value sought. And 104.62 *l.* divided by 11.7, the value of the joint lives A and B by Table 6th, quotes 8.942 *l.* for the same value in *annual* payments, the first to be made at the *end* of the year.

PROBLEM XIX.

To find the value of a given sum payable if A should be the *first* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given sum payable if C (the elder of B and C) should be living at the decease of A in case B is then dead.—Find next by Problem 16th the value of the given sum payable if C is living at the decease of A whether B is then living or not.—Subtract the former value from the latter, and the remainder will be the answer.

If

If the value is required in *annual* payments to be continued till the claim is either determined or becomes due, the preceding answer must be divided by an Annuity on the joint continuance of the three lives.

EXAMPLE.

Let the age of A be 20 years, that of B 30, and that of C 60. The sum 1000*l.*—Then by Example 1st, Problem 18th, the first value appears to be 18.75; and by Problem 16th the second value is 141.26. Their difference, therefore, or 122.51*l.* will be the value required.—And 122.51 being divided by 6.88, the value of the three joint lives by Problem 5th, the quotient 17.8*l.* is the same value in *annual* payments, the first to be made at the *end* of the year.

PROBLEM XX.

To find the value of a given sum payable if A should be the *second* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given sum payable if B should live to the decease of A, and C is then extinct. Find

L 3

also

also by the same Problem the value of the given sum payable if C should live to the decease of A, and B is then extinct. The sum of these two values will be the answer in *one* present payment.

To find the value in *annual* payments till the claim is either determined or becomes due; subtract the value of an Annuity on the three joint lives from the *sum* of the values of the two joint lives A and C, and the two joint lives A and B. Divide the *single* payment by this remainder, and the quotient will be the answer.

E X A M P L E.

Let the age of A be 20 years, that of B 30, and that of C 60.—The sum 1000 *l*.—By Example 3d, Problem 18th, the first value appears to be 104.62; and by Example 1st the latter value appears to be 18.75. Their sum, therefore, or 123.37 *l*. is the answer in *one present* payment.

The value of the joint lives A and B (by Table 6th) is 11.711, of the joint lives A and C (by the same Table) 7.967, and of the three joint lives A, B, and C (by Problem 5th) 6.882. This last value being subtracted from the sum of the two former values (or 19.678) leaves 12.796; and
123.37 *l*.

123.37 *l.* divided by this remainder quotes 9.641 *l.* for the value in *annual* payments, the first to be made at the *end* of the year.

PROBLEM XXI.

To find the value of a given sum payable at the death of A, if his life should be the *last* that fails of the three lives A, B, and C.

SOLUTION.

Find by Problem 18th the value of the given sum payable if C should live till the decease of A in case B should be then dead (C being supposed the elder of B and C.) Find next by Problem 17th the value of the given sum payable if A should die *after* B. Subtract the former value from the latter, and the remainder will be the answer in *one present* payment.

The value in *annual* payments till the claim is *determined* is found by the rule in Problem 20th.—And the same value till the claim *becomes due* (provided it should be determined in favour of the purchaser) is found by dividing the single payment by an Annuity on the life of A.

EXAMPLE.

Let the age of A be 60 years, that of B 20, and that of C 30. The sum 1000 *l.*— By Case 2d, Problem 18th, the first value will be found equal to 75.288, and by Problem 17th the latter value equal to 100.877; their difference, therefore, or 25.589 *l.* is the value required in *one present* payment.

By proceeding according to the rule in Problem 20th the difference between the three joint lives, (found by Problem 5th) and the *sum* of the values of the two joint lives A and C, and the two joint lives A and B (by Table 6th) will be found equal to 8.864. And 25.589 *l.* divided by this difference quotes 2.886 *l.* for the value in *annual* payments, to be made at the *end* of every year till the claim is *determined*.

The value of the single life A (by Table 5th) is 9.02, and 25.589 *l.* divided by this Annuity gives 2.836 *l.* for the *annual* payments to be made as above till the claim *becomes due*.

These, it may be observed, are the values agreeable to Mr. *De Moivre's* hypothesis, Had the respective ages of A, B, and C been

been 10, 10, and 70, the value of the Reversion, by the same hypothesis, in *one present* payment, would have been equal to 82.643*l.* and by the *Northampton* Table of Observations equal to 79.416*l.* *

P R O B L E M XXII.

To find the value of a given sum payable at the extinction of the lives A and B, should they be the *first* that will fail of the three lives A, B, and C. .

S O L U T I O N.

Find by Problem 18th the expectation of C on the contingency of A's surviving B, and also his expectation on the contingency of B's surviving A. The *sum* will be the value required in *one present* payment.

To obtain the same value in *annual* payments till the claim is either *determined* or *becomes due*; find the value of the two joint lives B and C, and also the value of the two joint lives A and C, from the *sum* of which subtract the value of the three joint lives; then divide the single payment by the remainder, and the quotient will be the answer.

* See the latter part of the Example to Problem 17th.

E X A M.

EXAMPLE.

Let the age of A be 20 years, that of B 60, and that of C 30. The sum 1000 *l.*—Here the first part of C's expectation appears from Example 3d, Problem 18th, (by substituting C for B and *vice versa*) to be 104.62 *l.*; and the latter part of his expectation is found by Case 2d (substituting C for B, A for C, and B for A) to be 75.288 *l.* Therefore 179.908 *l.* is the answer in *one present* payment.

Again.—The value of the joint lives B and C (by Table 6th) is 7.779, and of the joint lives A and C (by the same Table) 11.711; from the *sum* of which 6.882 (the value of the three joint lives by Problem 5th) being subtracted, we have 12.6 for the remainder: And 179.908 *l.* being divided by 12.6 quotes 14.278 *l.* for the value of the above expectation in *annual* payments to be made at the end of every year till the claim is determined or becomes due.

PROBLEM XXIII.

To find the present worth of an estate of a given yearly value, to be entered upon at the

the *decease** of A and B, should their lives be the *last* that shall fail of the three lives A, B, and C.

SOLUTION.

CASE 1st. *If either A or B are older than C.* Find by Problem 6th the value of an Annuity on the longest of three equal lives whose common age is that of the elder of the two lives A and B. Subtract this value from the perpetuity; divide the remainder by 3 and *reserve* the quotient.—Find next the value of 1*l.* payable at the end of as many years as are expressed by the complement, or twice the expectation, of the youngest of A and B; which being multiplied into half the perpetuity, let the product be subtracted from the *reserved* quotient. Then it will be, as the products of

* Care should be taken to distinguish between this and the question “for finding the value of an estate, or “equivalent sum, to be received *as soon as it is determined* whether A and B shall be the *last* that will fail “of the three lives A, B, and C.” This is the same with the 17th Problem and differs widely from the 23d; for it is evident that in the one the expectant is to become possessor of the Reversion on the extinction of the *joint* lives, but that in the other he cannot come into possession till the extinction of the *longest* of the lives.

the

the *complements* of C and the youngest of the two lives A and B, to the square of the *complement* of the oldest of them, so is the preceding remainder to a fourth proportional, which must be *reserved*.

Find next by Problem 4th the value of an Annuity on the life of the youngest of A and B for a term equal to the *difference* of the *complements* of those two lives, which subtract from the perpetuity and take half the remainder. Then it will be, as the *complement* of C is to the *complement* of the eldest of A and B, so is the said half remainder to a fourth proportional; which being multiplied into the value of 1 *l.* payable at the end of a term equal to the *complement* of the eldest of A and B, and added to the *reserved* proportional, the *sum* will be the number of years purchase required.

CASE 2d. *If C is the oldest of the three.* Find by the preceding rule the value of the estate to be entered upon at the decease of A and C, if B should be the first that fails. Find also the same value at the decease of B and C if A should be the first that fails. Add these two values to the value of an Annuity on the *longest* of the three lives A, B, and C (found by Problem 6th;) then subtract the *sum* from the perpetuity, and the remainder

mainder will be the number of years purchase required.

If in either of those two cases, the number of years purchase be multiplied by any given *sum*, and divided by the perpetuity increased by unity, the quotient will be the value of the Reversion of this *sum* *.

The above value in *annual* payments till the claim is *determined* may be obtained, by dividing the value in *one* payment by the three joint lives (found by Problem 5th.)

In order to obtain the value in *annual* payments till the money *becomes due*; that is, till the claim is determined, and afterwards till it becomes payable provided it should be determined in favour of the purchaser, divide the value in *one* payment by the value of an Annuity on the three *joint* lives added to the *difference* between the value of an Annuity on the life of C and on the *longest* of the three lives A, B, and C.

EXAMPLE FIRST.

Let the age of A be 20 years, that of B 60, and that of C 30. The estate, 40*l.* *per annum*, which is equivalent to the single

* See the demonstration of this Problem in note F, Appendix.

sum

sum of 1000 *l.*—This example, it is plain, belongs to Case 1st.—The value of the longest of three lives aged 60 is 12.987, which subtracted from 25 (the perpetuity) and the remainder divided by 3 becomes equal to 4.004. The value of 1 *l.* at the end of 66 years, or a term equal to the complement of A, is (by Table 1st) .075, which multiplied into half the perpetuity, or 12.5, is equal to .9375; and this subtracted from 4.004 leaves 3.0665. Therefore as 56 (the complement of C) multiplied into 66 (the complement of A.) or 3696, is to the square of 26 (the complement of B,) or 676, so is 3.0665 to .5607, the fourth proportional to be *reserved*.—Again. The value of an Annuity on the life of A for $(66 - 26 =)$ 40 years is 15.151, which subtracted from 25, and the remainder divided by 2, becomes equal to 4.924. Hence as 56 is to 26 so is 4.924 to 2.29; and this proportional multiplied by .3607, the value of 1 *l.* (by Table 1st) at the end of 26 years, is reduced to .825. Consequently 1.3857 is the number of years purchase required; which being multiplied by 1000 and divided by 26 quotes 53.296 *l.* for the value of the given *equivalent sum in one present payment*.

The

The value of the three *joint* lives is 6.882, and 53.296 *l.* divided by this quantity quotes 7.744 *l.* for the value of the above sum in *annual* payments till the claim is *determined*.

The value of the three *joint* lives being 6.882, the value of the *longest* of the three lives (by Problem 6th) 19.165, and the value of the life of C (by Table 5th) 14.684; the *difference* between the latter and the *sum* of the two former values will be 11.363, and 53.296 *l.* divided by this difference quotes 4.69 for the value in *annual* payments to be continued till the claim *becomes due* *.

Such are the values by Mr. *De Moivre's* hypothesis. Suppose now the value of the preceding Reversion in *one present* payment is required agreeable to the *Northampton* Table of Observations. By Scholium to Problem 6th it appears that the *exact* value of the longest of three lives aged 60 is 13.1013. By Table 4th, Appendix, the complements of the lives of A, B, and C

* If instead of making the *annual* payments at the *end* of every year, they are to be made at the *beginning* of each year, *unity* must be added to the divisors for determining those values;—that is, in the present case 53.296 *l.* must be divided either by 7.882 or 12.363.

are 66.38, 26.26, and 56.76 respectively. Therefore following the same steps as in the foregoing Example the *reserved* proportional will be found equal to .5565.—The value of an Annuity on the life of A for $(66.38 - 26.26 =)$ 40 years, by Problem 4th and the Table in page 64, is 15.2; consequently the number to be added to the *reserved* proportional is .805, which makes the value of the *estate* to be 1.3615 years purchase; hence the value of the given sum is 52.365 *l*.

EXAMPLE SECOND.

Let the age of A be 20 years, that of B 30, and that of C 60.—The sum 1000 *l*. This Example belongs to Case 2d; therefore the first value by substituting C for B and *vice versa*, appears by the preceding Example to be 1.3857; and by following the directions laid down in Case 1st (after substituting C for A and *vice versa*) the second value will be found equal to 1.2898. Their *sum* added to 19.165, the value of the longest of the three lives, and then subtracted from 25, the perpetuity, leaves 3.1595 for the number of years purchase required; and consequently the value of the given sum is 121.52 *l*.

The

The value of the three *joint* lives being 6.882, the above value in *annual* payments, to be made at the *end* of every year till the claim is *determined*, will be 17.657*l.*—And the *difference* between the life of C (by Table 5th) and the *sum* of the values of the three *joint* lives and the *longest* of the three lives being 17.03, the *annual* payments, to be made at the end of every year till the claim *becomes due*, will be 7.135 *l.*

PROBLEM XXIV.

To find the value of a given sum payable at the decease of A provided *both* or *either* of the two lives B and C are then living; that is, if the life of A should be the *first* or *second* that shall fail of the three lives A, B, and C.

SOLUTION.

Find by Problem 21st the value of the given sum payable if A should be the *last* that dies of the three lives: Subtract this from the value of the Reversion of the given sum after the decease of A (found by Problem 8th) and the remainder will be the answer in *one present* payment.

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The same value in *annual* payments, till the claim is either *determined* or *becomes due*, will be obtained by dividing the value in one payment by an Annuity on the three joint lives subtracted from the sum of the values of an Annuity on the joint lives of A and C and an Annuity on the joint lives of A and B.

EXAMPLE.

Suppose the age of A to be 20 years, that of B 30, and that of C 60. The sum 1000*l*. —The first value by Problem 21st may be found equal to 106.096 *l*. and the second value (by Problem 8th) equal to 350.346 *l*. therefore the value required in *one* payment, by the hypothesis, is 244.25 *l*.

The value of the three joint lives, by the same hypothesis and Problem 5th, is 6.882, which subtracted from 19.678, the *sum* of the values of the joint lives A and C, and the joint lives A and B (by Table 6th) leaves 12.796. Hence 244.25 divided by this remainder, quotes 19.088 *l*. for the value in *annual* payments to be made at the end of every year till the claim is either determined or becomes due.

PROBLEM XXV.

To find the value of a given sum payable at the decease of A if his life should *not* be the first that shall fail of the three lives A, B, and C; that is, if it should be the *second* or *third* that fails of those lives.

SOLUTION.

Find by Problem 19th the value of the given sum if A should be the *first* that dies: Subtract this value from the value of the Reversion of the given sum after A's life (found by Problem 8th,) and the remainder will be the answer in *one present* payment.

The value in *annual* payments till the claim is *determined*, is found by the rule in Problem 20th. And the value in those payments till the claim *becomes due* is found by dividing the single payment by the value of an Annuity on the life of A.

EXAMPLE.

Let the age of A be 20 years, that of B 30, and that of C 60. The sum 1000 *l.*—The value on the first contingency (by Problem 19th) will be found equal to 122.51 *l.* and

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the value of the Reversion after A's life (by Problem 8th) is 350.346 *l.*; consequently 227.836 *l.* will be the value required, by the hypothesis, in *one* payment.

227.836 *l.* divided by 12.796 (found by the rule in Problem 20th) quotes 17.805 *l.* for the value in *annual* payments to be made at the end of every year till the claim is *determined*. And 227.836 *l.* divided by 15.89 (the value of an Annuity on the life of A by Table 5th) gives 14.33 *l.* for the value in those payments, to be made in the same manner till the claim *becomes due*.

PROBLEM XXVI.

To find the value of a given sum payable at the decease of A, if his life should *not* be the second that fails; that is, if it should be the *first* or *last* that shall fail of the three lives A, B, and C.

SOLUTION.

Find by Problem 20th the value of the given sum provided A should be the *second* that dies: Subtract this value from the value of the Reversion after A's life (found by Problem 8th) and the remainder will be the answer in *one present* payment.

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The value in *annual* payments till the claim is *determined* is found as in Problem 20th.—The value in *annual* payments till the claim *becomes due* (provided it should be determined in favour of the purchaser) is the value in *one* payment divided by the value of the single life A.

EXAMPLE.

Suppose the age of A 20, that of B 30, and that of C 60 years. The sum 1000 *l.* The first value according to Mr. *De Moivre's* hypothesis appears to be 123.34 *l.* and the second value 350.346 *l.* Their difference, therefore, or 227.006 *l.* is the value required in *one present* payment;—which being divided by 12.796 (found by the rule in Problem 20th) quotes 17.74 *l.* for the value in annual payments to be made at the end of every year till the claim is determined: and being divided by 15.89 (the value of an Annuity on the life of A by Table 5th) quotes 14.286 *l.* for the value in those payments to be made in the same manner till the claim *becomes due*, provided it should be determined in favour of the purchaser.

PROBLEM XXVII.

To find the value of an Annuity on the life of B after A, on the particular condition that A's life when it fails, shall fail before the life of C.

SOLUTION.

CASE 1st. *When B is oldest; or when B is older than C, and C is older than A.*—From the value of an Annuity on the two joint lives B and C, subtract the value of an Annuity on the three joint lives A, B, and C; and reserve the remainder.

Find by the rule in page 55 the value of an Annuity on the longest of the two lives A and C, and by Problem 6th, or its Scholium, the value of an Annuity on the longest of the three lives A, B, and C; then will half the difference between these two last values added to the reserved remainder be the number of years purchase required.

CASE 2d. *When B is younger than C, and C older than A.* Find, as in the preceding Case, the difference between the values of the two joint lives B and C, and the three joint lives A, B, and C, and let it be reserved.—Find next the value of an Annuity

nunity on the longest of two equal lives whose common age is that of C, and subtract it from the value of an Annuity on the longest of three lives, the common age of two of whom is that of C, and the age of the third that of B; multiply this remainder by the expectation of C, divide the product by twice the expectation of A, and the quotient added to the remainder, *reserved* above, will be the number of years purchase required.

CASE 3d. *When B is younger than C, and C younger than A.* Find the value as in Case 2d, upon the supposition that C should die before A. Add this value to that of an Annuity on the three joint lives; let the sum be subtracted from the value of an Annuity on the single life B, and the remainder will give the number of years purchase required.

CASE 4th. *When B is older than C, and A older than B.* This is answered in the same manner with Case 3d, by first finding the value of an Annuity by Case 2d on the life of B after C on condition that C dies before A, and then subtracting this value, added to the value of the three joint lives, from the single life B*.

* See the demonstration of this Problem in note G, Appendix,

EXAMPLE FIRST.

Let the age of B be 60 years, that of C 30, and that of A 20.—The value of the joint lives of B and C (by Table 6th) is 7.779, and the value of the three joint lives (by Problem 5th and the hypothesis) is 6.882; therefore the remainder to be reserved is .897.

The value of the longest of the three lives (by Problem 6th and the hypothesis) is 19.165, and that of the longest of the two lives A and C (by the rule in page 55) is 18.864; consequently half their difference is .15, which added to .897 gives 1.047 for the number of years purchase required*.

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* The answers in this and the following Examples are a little *above* the truth, owing to the value of the three joint lives, and that of the longest of the three lives being given by Prob. 5th and 6th, the one too high and the other too low. These deviations arise from the rules in those Problems being only *approximations* to the real values.—But they may always be used with safety, provided we approximate to the values of single and joint lives (by Prob. 2d and 3d) from a *given rate of mortality* when the ages exceed 60; Mr. *De Moivre's* hypothesis, (as I have already observed in the Scholium to the sixth Problem) producing errors too considerable after this period.—Having regard to this precaution, we shall seldom or ever deviate from the truth more than $\frac{1}{10}$ th of a year's purchase.

—I should

The value of this Reversion by the *Northampton* Table, and approximating to the values of the single and joint lives by the rules in Prob. 2d and 3d, is .88.

EXAMPLE SECOND.

Let the age of B be 20 years, that of C 60, and that of A 30.—The value of the joint lives B and C (by Table 6th) being 7.967, and the value of the three joint lives (by Problem 5th) being 6.882, the difference

—I should not have taken notice of these circumstances, had it not been that, in the present Example, the value of the life of B after A, though restrained to the contingency of C's surviving A, appears to be almost equal to the value of the life of B after A without any such restriction; from whence the reader might be led to conclude that the rule is wrong.—Now it is evident that when the ages of the lives are respectively as in this Case, there cannot be much difference between them.—If the values, however, of the three joint lives and the longest of the three lives be computed in exact agreement to the hypothesis by Problem 1st, Corollary 5th, and Problem 2d, Corollary 2d, in *Simpson's Doctrine of Annuities*, they will be found equal to 6.949 and 19.084, which will reduce the value of the Reversion to .94, or nearly $\frac{1}{10}$ th of a year's purchase below the value given in the Example.—This reduced value may be proved to be accurately true from other principles.—See *Simpson's Select Exercises*, Problem 35th, where a different solution is given of this Problem, but which agrees in the result exactly with the above.

to be reserved is 1.085.—The value of an Annuity on the longest of the three lives B, C, and C (by Problem 6th and *De Moivre's* hypothesis) is 17.445, and on the longest of the two lives, whose common age is that of C, (by the rule in page 55 and the same hypothesis) 11.683 : The latter value subtracted from the former leaves 5.762; which multiplied by 13 (the expectation of C) and then divided by 56 (twice the expectation of A) will be equal to 1.337; and this quantity added to 1.085, reserved above, gives 2.422 for the answer in this case.

The value of this Reversion by the *Northampton* Table and the approximations in Problems 2d and 3d, is 2.429.

EXAMPLE THIRD.

Let the age of B be 20, that of C 30, and that of A 60 years. The value of the Reversion (by Case 2d) upon the supposition that C should die before A is 2.422. The value of the three joint lives is 6.882; and both these added together and then subtracted from 15.891 (the value of the life of B by Table 5th) will leave 6.587 for the number of years purchase required agreeable to Mr. *De Moivre's* hypothesis.

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The same value by the *Northampton* Table and the approximations in Problems 2d and 3d is 6.5.

EXAMPLE FOURTH.

Let the age of B be 30, that of C 20, and that of A 60 years: Then substituting A for C and *vice versa*, the first value by proceeding as in Case 2d will be found by Mr. *De Moivre's* hypothesis equal to 1.85, which added to 6.882, the value of the three joint lives by the same hypothesis, and the sum subtracted from 14.684 (the value of the life of B by Table 5th) gives 5.952 for the answer.

By the *Northampton* Table and the approximations in Problems 2d and 3d, this Reversion will be found worth 5.985 years purchase.

Had the age of all the three lives been 60, the value by Mr. *De Moivre's* hypothesis would have been equal to 1.992; by the *Northampton* Table and the approximations just mentioned, it would have been equal to 2.055; and by the same Table of Observations (taking the values of the single and joint lives from Table 7th, Appendix, and the two Tables in pag. 64 and 74) the *exact* value would have been equal to 2.118.

COROL-

COROLLARY.

From the preceding Solution the value of the life of B after A, provided A should die *after* C, may be easily determined, the rule being as follows :

“ Find by this Problem the value of an
 “ Annuity on the life of B after A, pro-
 “ vided A should die *before* C. Find next
 “ the value of the joint lives of A and B.
 “ Then will the *sum* of these two values
 “ subtracted from the value of the life of
 “ B, be the number of years purchase re-
 “ quired.”

EXAMPLE.

Let the age of B be 20, that of A 30, and that of C 60 years.—By Example 2d of this Problem it appears that the first value, according to Mr. *De Moivre's* hypothesis, is 2.422, and by Table 6th that the second value is 11.711. Their *sum*, therefore, or 14.133, subtracted from 15.891 (the value of the life of B by Table 5th) leaves 1.758 for the answer.

The same value by the *Northampton* Table and the approximations in Problems 2d and 3d is 1.711.

PROB-

PROBLEM XXVIII.

To find the present worth of an estate of a given yearly value to be entered upon at the decease of A and B, provided C should survive one life in particular (A.)

SOLUTION.

Find the value of the given estate by Problem 16th if C should live till the decease of A.—Find next by Problem 27th the value of the life of B after A, provided C should survive A. Subtract the last value from the former, and the remainder will be the answer.

The value in *annual* payments till the claim is *determined*, is the single payment divided by the value of the joint lives A and C.—And the value in *annual* payments till the claim *becomes due* (provided it should be determined in favour of the purchaser) is the single payment divided by the value of the joint lives A and C, increased by the value of the life of B after A, provided C shall have survived A (found by Problem 27th.)

If the value of the *estate* in *one* payment be divided by *1 l.* increased by its interest for

for a year, the quotient will be the value of an equivalent *sum*.

EXAMPLE.

Let the age of A be 30 years, that of B 60, and that of C 20.—The estate 40 *l. per annum*.

The first value by Problem 16th, agreeable to the hypothesis, is 7.427 years purchase. The second value by Problem 27th and the hypothesis, is 1.235 years purchase, which being subtracted from 7.427, and the remainder, or 6.192, multiplied by 40, we have 247.68 *l.* for the answer in *one present payment*.

And 247.68 *l.* divided by 1.04 (or 1 *l.* increased by its interest for a year) quotes 238.15 *l.* for the value of an equivalent *sum* (or of 1000 *l.*) in *one payment*.

The value of the joint lives of A and C (by Table 6th) is 11.711, and 238.15 *l.* divided by this quantity quotes 20.335 *l.* for the value of the above sum in *annual payments* to be made at the end of every year till the claim is *determined*.—But if those payments are to be continued in the same manner till the claim *becomes due*, the divisor 11.711 must be increased by 1.235 (found

(found by Problem 27th) which will consequently reduce them to 18.395 *l*.

The value of the *estate* in one payment by the *Northampton* Table, and the approximations in Problems 2d and 3d, may be found equal to 248.52 *l*. and of the *equivalent* sum equal to 238.96 *l*.

PROBLEM XXIX.

To find the value of a given sum payable at the death of C, provided A should be the *first*, B the *second*, and C the *third* that fails of the three lives A, B, and C.

SOLUTION.

CASE 1st. *If the life of C is the oldest of the three.* From the perpetuity subtract the value of an Annuity upon the longest of three equal lives whose common age is that of C, and take $\frac{1}{2}$ th the remainder.—Then it will be, as the product of the complements * of the two other lives is to the square of the complement of C, so is the said $\frac{1}{2}$ th remainder to a fourth proportional, which will be the number of years purchase an equivalent *estate* is worth; and this being

* The complement is equal to twice the expectation.

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multiplied into the given sum, and the product divided by the perpetuity increased by unity, the *quotient* will be the answer.

CASE 2d. *If the life of B is the oldest.* From the perpetuity subtract the value of an Annuity upon the longest of three equal lives whose common age is that of B, and take $\frac{1}{6}$ th the remainder. — Multiply half the perpetuity by 1 *l.* payable at the end of as many years as are expressed by the complement of C. Then it will be, as the complements of A and C multiplied into each other are to the square of the complement of B, so is the *difference* between the product just mentioned and the $\frac{1}{6}$ th remainder to a fourth proportional, which must be *reserved*. Find next by Problem 4th the value of an Annuity on the life of C for a term equal to the difference between his complement and that of B, which subtract from the perpetuity and take $\frac{1}{2}$ the remainder. Then it will be, as the expectation of A is to the expectation of B, so is the said remainder to a fourth proportional; which being multiplied into the value of 1 *l.* payable at the end of a term equal to the complement of B, and added to the *reserved* proportional, the *sum* will be the number of

of years purchase an equivalent *estate* is worth, when the $\frac{1}{6}$ th remainder is *greater* than the product; but if *less*, the reserved proportional must be subtracted, and the *remainder* will be the number of years purchase the said *estate* is worth.—The value of the given *sum* is obtained from hence, in this and the following Cases, as in Case 1st.

CASE 3d. *If A is the oldest and B the youngest.* From the perpetuity subtract the value of an Annuity upon the longest of three lives whose common age is that of A, and take $\frac{1}{6}$ th the remainder.—Then it will be, as the complements of B and C multiplied into each other are to the square of the complement of A, so is the said $\frac{1}{6}$ th remainder to a fourth proportional, which must be *reserved*.—Find the value of an Annuity on the life of C for as many years as are expressed by the difference between his complement and that of A, which being subtracted from the perpetuity, let the remainder be multiplied into the expectation of A.—Find next by Problem 4th, Corollary 1st, the value of an Annuity during the term just mentioned upon the longest of two equal lives whose common age is that of C, which subtract from the perpetuity, and multiply

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the remainder by the expectation of C. Add these two products together; then it will be, as the complement of B is to the value of 1 *l.* payable at the end of as many years as are expressed by the complement of A, so is the *sum* of the two products to a fourth proportional, which must also be *reserved*.—Lastly, multiply the perpetuity into the expectation of A, and also into the value of 1 *l.* payable at the end of as many years as are expressed by the complement of C: divide the product by the expectation of B; then the quotient subtracted from the sum of the two *reserved* proportionals will give the number of years purchase required.

CASE 4th. *If A is oldest and C youngest.* Find by Case 3d (after substituting there B for C, and C for B) the value of the given estate upon the supposition that B is the youngest.—Then find the value of an Annuity on the life of C for as many years as his complement exceeds that of B: Subtract this value from the perpetuity, let the remainder be multiplied into 1 *l.* payable at the end of a term equal to the complement of B; add this product to the value found by Case 3d, and *reserve* the sum.—Lastly, multiply the perpetuity into the value of 1 *l.* payable at the end of as many years as are
expres-

expressed by the complement of C, and also into the expectation of B; divide the product by the expectation of C, and this quotient being subtracted from the *reserved* sum, the remainder will be the number of years purchase required.

In either of those cases the value in *annual* payments, to be continued till the claim is *determined*, may be obtained by dividing the value in *one* payment by an Annuity on the joint lives of B and C.

If the *annual* payments are to be continued till the claim *becomes due*; that is, till the claim is determined, and afterwards till it becomes payable provided it should be determined in favour of the purchaser, divide the value in *one* payment by the value of the joint lives B and C added to the value (found by Corollary to Problem 27th) of an Annuity on the life of C after B provided B survives A.

EXAMPLE FIRST*.

Let the age of A be 20 years, that of B 30, and that of C 60.—The sum 1000 l.—
This

* The answers to these four Examples are computed agreeable to Mr. *De Moivre's* hypothesis, and are compared in Note H of the Appendix not only with the

This Example, it is evident, belongs to Case 1st.—The value of an Annuity upon the longest of three equal lives aged 60 (by Problem 6th) is 12.987, which subtracted from 25 (the perpetuity) and the remainder divided by 6 becomes equal to 2.002. Therefore as 56 (the complement of B) multiplied into 66 (the complement of A,) or 3696, is to 676 (the square of 26, the complement of C,) so is 2.002 to .3661, which multiplied into 1000 (the given sum) and divided by 26 (the perpetuity increased by unity) quotes 14.08 l. for the answer in *one present* payment.

The value of the joint lives B and C (by Table 6th) is 7.779; hence 14.08 l. divided by this value gives 1.81 l. for the *annual* payments till the claim is determined*.—

The value of the life of C after B, provided

same values in *exact* conformity to the *Northampton* Table of Observations, and also with the values calculated from this Table and the rules in Problems 2d and 3d for approximating to the single and joint lives; but likewise with the values as they are derived from a different investigation of the Problem.

* The *annual* payments in those Examples are supposed to be made at the *end* of every year. If they are to be made at the *beginning* of each year, unity must be added to the number by which the single payment is divided.

B survives A, may be found by Corollary to Problem 27th equal to .111, which being added to 7.779 and 14.08/ divided by their sum, we shall have 1.784/ for the value in *annual* payments till the claim becomes due.

EXAMPLE SECOND.

Let the age of A be 20, that of B 60, and that of C 30 years.—The sum 1000/. —By following the steps directed in the first part of Case 2d the $\frac{1}{6}$ th remainder will be found equal to 2.002. Half the perpetuity, or 12.5, multiplied into .111, the value of 1/ (by Table 1st) at the end of 56 years, is equal to 1.3875. As, therefore, 3696 is to 676 so is .6145 (the difference between 2.002 and 1.3875) to .1123, which must be *reserved*.—The value of an Annuity on the life of C for $(56 - 26 =)$ 30 years will be found (by Problem 4th) equal to 13.393, which subtracted from 25 and divided by 2 quotes 5.803. Hence as 33 (the expectation of A) is to 13 (the expectation of B) so is 5.803 to 2.286, which multiplied into .3607, the value of 1/ (by Table 1st) at the end of 26 years, is reduced to .8246; and this added to .1123, *reserved* above,

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gives

This Example, it is evident, belongs to Case 1st.—The value of an Annuity upon the longest of three equal lives aged 60 (by Problem 6th) is 12.987, which subtracted from 25 (the perpetuity) and the remainder divided by 6 becomes equal to 2.002. Therefore as 56 (the complement of B) multiplied into 66 (the complement of A,) or 3696, is to 676 (the square of 26, the complement of C,) so is 2.002 to .3661, which multiplied into 1000 (the given sum) and divided by 26 (the perpetuity increased by unity) quotes 14.08/. for the answer in *one present* payment.

The value of the joint lives B and C (by Table 6th) is 7.779; hence 14.08/. divided by this value gives 1.81 l. for the *annual* payments till the claim is determined*.—The value of the life of C after B, provided

same values in *exact* conformity to the *Northampton* Table of Observations, and also with the values calculated from this Table and the rules in Problems 2d and 3d for approximating to the single and joint lives; but likewise with the values as they are derived from a different investigation of the Problem.

* The *annual* payments in those Examples are supposed to be made at the *end* of every year. If they are to be made at the *beginning* of each year, unity must be added to the number by which the single payment is divided.

B survives A, may be found by Corollary to Problem 27th equal to .111, which being added to 7.779 and 14.08/ divided by their sum, we shall have 1.784/ for the value in *annual* payments till the claim becomes *due*.

EXAMPLE SECOND.

Let the age of A be 20, that of B 60, and that of C 30 years.—The sum 1000/.—By following the steps directed in the first part of Case 2d the $\frac{1}{6}$ th remainder will be found equal to 2.002. Half the perpetuity, or 12.5, multiplied into .111, the value of 1/ (by Table 1st) at the end of 56 years, is equal to 1.3875. As, therefore, 3696 is to 676 so is .6145 (the difference between 2.002 and 1.3875) to .1123, which must be *reserved*.—The value of an Annuity on the life of C for $(56 - 26 =)$ 30 years will be found (by Problem 4th) equal to 13.393, which subtracted from 25 and divided by 2 quotes 5.803. Hence as 33 (the expectation of A) is to 13 (the expectation of B) so is 5.803 to 2.286, which multiplied into .3607, the value of 1/ (by Table 1st) at the end of 26 years, is reduced to .8246; and this added to .1123, *reserved* above,

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gives .9369 for the number of years purchase an estate is worth. Consequently the value of the proposed sum in *one present* payment is 36.035 *l.*; which divided by 7.779 (the value of the joint lives of B and C by Table 6th) quotes 4.632 *l.* for the value in *annual* payments till the claim is *determined*.—And 36.035 *l.* divided by 7.779 added to .953 (the value of the life of C after B provided B survives A by Corollary to Problem 27th) quotes 4.127 *l.* for the annual payments till the claim *becomes due*.

EXAMPLE THIRD.

Let the age of A be 60, that of B 20, and that of C 30 years. The sum 1000 *l.*—By proceeding agreeable to the Solution in Case 3d, the first proportional to be *reserved* appears to be .3661.—The value of C's life for 30 years is 13.393, which subtracted from 25, and then multiplied by 13 (the expectation of A) is equal to 150.891.—The value of the longest of two equal lives aged 30 for 30 years will be found (by Corollary 1st to Problem 4th) equal to 16.025, which subtracted from 25 and multiplied by 28 (the expectation of C) is equal to 251.3.—Therefore, as 66 (the complement of B) is to .3607 (the value of 1 *l.* by Table 1st,

1st, at the end of 26 years) so is 402.191 (the sum of 150.891 and 251.3) to 2.198, which is the second proportional to be reserved. Lastly, 25 multiplied into .111 (the value of 1l. at the end of 56 years) and also into 13 (the expectation of A) produces 36.075, which divided by 33 (the expectation of B) is reduced to 1.0931; and this quotient subtracted from 2.5641 (the sum of the two reserved proportionals) leaves 1.471 for the number of years purchase an estate is worth. Consequently 56.577 l. is the value of the given sum in one present payment; which being divided by 11.7, the value of the joint lives B and C by Table 6th, quotes 4.835 l. for the value in annual payments till the claim is determined.—And 56.577 l. divided by 11.7 added to 1.12 (the value of the life of C after B provided B survives A by Corollary to Problem 27th) quotes 4.413 l. for the annual payments till the claim becomes due.

EXAMPLE FOURTH.

Let the age of A be 60, that of B 30, and that of C 20 years. The sum 1000 l.—By substituting C for B and *vice versa*, the present Example belongs to Case 3d, and the value of an estate on such a supposition

N 4

appears

appears to be 1.471 years purchase.—Let now the symbols represent the ages as above, and it is converted into an example of the fourth Case.—The value of an Annuity (by Problem 4th) on the life of C for $(66 - 56 =)$ 10 years is 7.474, which subtracted from 25, and multiplied into .111 (the value of 1 *l.* by Table 1st, at the end of 56 years) becomes equal to 1.9454; therefore the sum to be *reserved* is 3.4164.—The value of 1 *l.* at the end of 66 years (by Table 1st) is .075, which multiplied into 25 and also into 28 (the expectation of B) produces 54.5; and this divided by 33 (the expectation of C) quotes 1.6515. Consequently 1.7649 is the number of years purchase an *estate* is worth, and 67.881 *l.* is the value of the given sum in *one present* payment.

If 67.881 *l.* be divided by 11.7, the value of the joint lives B and C by Table 6th, we shall have 5.802 *l.* for the value in *annual* payments till the claim is *determined*.

And if it be divided by 11.7 added to 1.76 (the value of the life of C after B provided B survives A by Corollary to Problem 27th) we shall have 5.043 *l.* for the value in *annual* payments till the claim *becomes due*.

PROB.

PROBLEM XXX.

B if he survives A in a given number of years (not exceeding the complement of the oldest of the three lives,) is to receive a given sum, in case C is then extinct; to determine the value of his expectation in present money.

SOLUTION.

Divide the sum of the decrements in the Table of Observations from the age of C for the given term by the term; and by this quotient divide the number of the living in the Table at the age of C; and this second quotient call the *complement* of the life of C *for the given term*.—Find in the same manner the complement of the life of A for the given term.

From the value of an Annuity on the life of B for the given term (found by Problem 4th) subtract the value of the joint lives of B and C for the same term (found by Corollary 1st to Problem 4th.) Multiply the remainder by the given sum, and divide the product by the complement of A *for the given term*, reserving the quotient.—Multiply the given sum by the value of an Annuity

nuity on the life of B for the given term and divide the product by double the complement of A *for the given term* multiplied into the complement of C *for the given term*; and the quotient subtracted from the last reserved quotient will be the answer*.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50, the term 16 years, the sum 1000*l.* and the Table of Observations the *Northampton* Table, or Table 3d, Appendix. The sum of the decrements in the Table for 16 years from 50, or the age of C, is 129, which divided by 16 (the given term) quotes 8.06. The number living at 50 is 284, which, divided by 8.06, gives 35.23 for the *complement* of the life of C for the given term.—In like manner it may be found that the *complement* of A for the given term is 45.

By Problem 2d the value of the life of B is 14.79, and of a life 16 years older 12.088; and consequently (by Problem 4th) the value of the life of B for 16 years is 10.086.—In the same manner, the value of the joint lives of B and C (by Problem 3d and Co-

* See what is said of the Solution of this and the following Problems in note I. Appendix,

rollary 1st to Problem 4th) for 16 years may be found to be 7.990.—The difference between these two values (or 2.096) multiplied into 1000 and divided by 45 (the complement of A for 16 years) gives 46.577 for the quotient to be *reserved*.—The sum 1000 *l.* multiplied by 10.086, and the product divided by twice the product of 45 into 35.23 gives .318; which subtracted from 46.577 leaves 46.259 *l.* the answer *.

N. B. In

* The solutions of this and all the following Problems are confined to a term of years not exceeding the complement, for the whole duration, of the oldest life. These, however, are abundantly sufficient, as it is never likely to happen that the values will be required for a longer term; and the solutions become so extremely complicated when they are extended further, that I have entirely omitted them.

The values, likewise, are only given in *one* present payment, being apprehensive of rendering the work too tedious by giving them in *annual* payments; especially as they may be so easily obtained from the solutions of the corresponding values for the *whole continuance* of the lives, where no other variation is necessary, than to find the values of the Annuities (by which the single payments are to be divided) for the *given term*, instead of finding them for the *whole duration* of life.—Thus in the present case; it appears from the solution of Problem 18th that the value in *annual* payments is found by dividing the single payment by an Annuity on the joint lives of A and B for 16 years. The value of this Annuity (by Corollary 1st to Problem 4th and the *Northampton Table* of

N. B. In this Example the values of the single and joint lives have been calculated by the rules in Problems 2d and 3d, taking the complements of the lives in strict agreement to the *Northampton Table*; or 56.76 and 35.88 for the ages of 30 and 50, and 39.74 and 20.54 for the ages of 46 and 66.

PROBLEM XXXI.

To find the value of a given sum payable if A should be the *first* that fails of the three lives A, B, and C, in a given number of years, not exceeding the complement of the oldest life.

SOLUTION.

Find first, in the manner directed in the last Problem, the *complements* of the lives of A, B, and C for the *given term*.—Then, from the perpetuity subtract the value of the longest of two equal lives whose common complement, for their whole duration, is the given term (found agreeable to the hy-

of Observations) is 8.476; therefore the value of the given sum in *annual* payments (to be made at the end of each year) is 5.457 *l*.

In general, the values of those Reversions for terms are so inconsiderable in *single* payments, that it can hardly be worth while to diminish them by *annual* ones.

pothesis

pothesis of an equal decrement.) Multiply *half* the remainder into the sum of the complements of B and C for the given term, and *reserve* the product.—Subtract the value (found by Problem 6th and agreeable to the hypothesis of an equal decrement) of the longest of three equal lives, whose common complement for the whole duration is the given term, from the perpetuity.—Multiply *one third* of the remainder into the given term, and let this product also be *reserved*.—Then say, as the complements for the given term of A, B, and C multiplied into each other are to the square of the given term, so is the *difference* between the two *reserved* products to a fourth proportional; which being multiplied into the given sum, divide the product by the perpetuity increased by unity, and *reserve* the quotient.—Find by Problem 9th the value of the given sum at the death of A, should that happen in the given term; subtract the *reserved* quotient from this value, and the remainder will be the answer.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50.—The term 16 years, and the proposed sum 1000 l.—By the rule in
page

page 55 the value of the longest of two equal lives whose complement is 16, (or whose age by *De Moivre's* hypothesis is 70) is 8.026; which subtracted from 25 (the perpetuity) and half the remainder, or 8.487, multiplied into 80.23 (the sum of the complements of B and C for the given term by Table 3d) gives 680.91 for the product to be *reserved*.—By Problem 6th and Tables 5th and 6th, the value of the longest of three equal lives whose common complement is 16, or age 70, is 8.971. This being subtracted from 25 (the perpetuity) and divided by 3 quotes 5.343, which multiplied into 16 gives 85.488 for the second product to be *reserved*.—Hence, as 93.504 * (the product of the three complements of A, B, and C for the given term) is to 256 (the square of 16) so is 595.43 (the difference between the two *reserved* products) to 1.630: which being multiplied into 1000, and divided by 26 (the perpetuity increased by unity) gives 62.70 for the quotient to be *reserved*.—By Problem 9th the value, by the *Northampton* Table, of the given sum at the death of A, should that happen in 16

* The complements of the lives of A, B, and C for 16 years are (by Table 3d) respectively equal to 58.98, —45,—and 35.23.

years,

years, is 195.7, which being lessened by 62.70, we have 133 *l.* for the value required in *one present* payment.

SCHOLIUM.

In finding the values of the longest of two or three equal lives, or of three joint lives, it is necessary to be possessed of the values of single lives and *two* joint lives; and these values, whenever necessary to be obtained for that purpose in solving the preceding Problem and all that follow, should be taken either from Tables of the values of single and joint lives by Mr. *De Moivre's* hypothesis, or computed agreeable to the hypothesis of an equal decrement, taking the term for the complement, by the rules in Problems 2d and 3d.—When not necessary to be obtained for the purpose of finding the values of two or three *equal* lives in solving these Problems, they should be taken agreeable to the Table of Observations used (or as nearly so as possible from the rules in Problems 2d and 3d) taking double the expectation (found by Problem 1st) for the complement.

PROB-

PROBLEM XXXII.

To find the value of a given sum payable at the death of A, should his life be the *second* that fails of the three lives A, B, and C, in a given number of years, not exceeding the complement of the oldest life.

SOLUTION.

First find the complements of the lives of A, B, and C for the given term as directed in Problem 30th.—From the perpetuity subtract the value of an Annuity on the longest of two equal lives whose common complement for their whole duration is the given term. Multiply *half* the remainder into the sum of the complements of B and C (just found) and *reserve* the product.—Subtract the value of an Annuity on the longest of three equal lives, whose common complement for their whole duration is the given term, from the perpetuity: multiply *one third* of the remainder into twice the given term, and let this product also be *reserved*.—Then say, as the complements for the given term of the three lives A, B, and C, multiplied by one another, are to the square of the given term, so is the *difference* of the

two

two *reserved* products to a fourth proportional, which being multiplied into the given sum, and divided by the perpetuity increased by unity, will give the value sought.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50. The term 16 years, and the proposed sum 1000*l*.—By proceeding in the same manner as in the Example to Problem 31st, the first product to be *reserved* will be found equal to 680.91, and the second product to be *reserved* equal to 170.976. Therefore as 93504, the three complements for the given term (by Table 3d) multiplied into each other, is to 256 (the square of 16) so is 509.934 (the *difference* of the two *reserved* products) to 1.396; which being multiplied into 1000, and then divided by 26 (the perpetuity increased by unity) gives 53.7*l*. for the answer in *one present* payment.

PROBLEM XXXIII.

To find the value of a given sum payable at the death of A, should his life be the *last* that fails of the three lives A, B, and C,

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in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find the complements of the lives of A, B, and C for the given term as directed in Problem 30th.—From the perpetuity subtract the value of an Annuity on the longest of three equal lives whose common complement for their whole duration is the given term, and take *one third* of the remainder: Then it will be, as the product arising from the complements, for the given term, of A, B, and C multiplied into each other, is to the cube of the given term, so is the *one third* remainder to a fourth proportional; which being multiplied into the given sum, and then divided by the perpetuity increased by unity, will give the value required.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50. The term 16 years. The sum 1000 *l*.—By Problem 6th and *De Moivre's* hypothesis, the value of three equal lives aged 70 is 8.971, which subtracted from 25 (the perpetuity) and the remainder divided

divided by 3 gives 5.343. Therefore as 93504 (the product of the complements of A, B, and C for the given term by Table 3d) is to 4096 (the cube of 16) so is 5.343 to .234. Consequently 94 is the value in *one present payment*.

PROBLEM XXXIV.

To find the value of a given sum payable at the extinction of the lives of A and B, should they be the *first* that will fail of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find the complements of the lives of A, B and C for the given term as directed in Problem 30th. Find next by the rule in page 55 the value of the longest of two equal lives whose common complement for their whole duration is the given term. Subtract this value from the perpetuity; then it will be, as the product, arising from the complements of the three lives for the given term multiplied by one another, is to this remainder, so is the square of the given term to a fourth proportional, which must

be *reserved*.—Again; find by Problem 6th the value of the longest of three equal lives whose common complement for the whole duration is the given term. Subtract this value from the perpetuity, and take *two thirds* of the remainder. Then say, as the product, arising from the three complements for the given term multiplied into each other, is to the *two third* remainder, so is the cube of the given term to a fourth proportional, which being subtracted from the *reserved* proportional, let the remainder be multiplied into the given sum; and this product, divided by the perpetuity increased by unity, will give the value required.

EXAMPLE.

Let the age of A be 30, that of B 40, and that of C 50.—The term 16 years, and the proposed sum 1000 *l*.—The value of two equal lives aged 70 (or whose common complement, by the hypothesis, for the whole duration is 16) may be found equal to 8.026, which being subtracted from 25 (the perpetuity) leaves 16.974. Therefore as 2654 (the product of the complements of A and B for the given term by Table 3d) is to 16.974, so is 256 (the square of 16) to 1.633, the proportional to be *reserved*.—

The value of the longest of three equal lives aged 70 is 8.971, which subtracted from 25, and the remainder multiplied into $\frac{2}{3}$, gives 10.686. Hence as 93504 (the product of the three complements for the given term into each other) is to 10.686, so is 4096 (the cube of 16) to .468.—And .468 subtracted from 1.633 (the *reserved* proportional) leaves 1.165; which being multiplied into 1000, and then divided by 26, quotes 44.808%. for the answer in *one present* payment.

PROBLEM XXXV.

To find the value of a given sum payable at the death of A and B, should their lives be the *last* that will fail of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

This Problem restricted, as above, to a number of years not exceeding the complement of the oldest life, is answered in the same manner with Problem 33d.

PROBLEM XXXVI.

To find the value of a given sum payable at the death of A, should his life be the *first* or *second* that shall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find by Problem 33d the value of the given sum payable if A should be the *last* that dies of the three lives in the given term. Subtract this from the value of the given sum (found by Problem 9th) payable at the death of A should that happen in the given term; and the remainder will be the answer.

EXAMPLE.

Suppose the age of A to be 30, that of B 25, and that of C 15.—The term 10 years, and the proposed sum 1000*l*.—The first value by Problem 33d may be found equal to .815, and the second value by Problem 9th and the *Northampton* Table, equal to 130.115. Their difference therefore, or 129.3*l*, is the answer in *one present* payment.

PROBLEM XXXVII.

To find the value of a given sum payable at the death of A, if his life should *not* be the second that fails; that is, if it should be the *first* or *last* that shall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find by Problem 32d the value of the given sum provided A should be the *second* that dies in the given term; subtract this value from the value of the given sum (by Problem 9th) payable at the death of A should that happen in the given time, and the remainder will be the answer.

EXAMPLE.

Let the age of A be 30, that of B 25, and that of C 15. The term 10 years, and the proposed sum 1000*l*.—The first value (by Problem 32d) may be found equal to 16.065; and the second value (by Problem 9th and the *Northampton Table*) equal to 130.115. Consequently 114.05*l*. is the answer in *one present* payment.

PROBLEM XXXVIII.

To find the value of a given sum payable at the death of A, if his life should *not* be the first that fails; that is, if it should be the *second* or *third* that shall fail of the three lives A, B, and C in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find by Problem 31st the value of the given sum if A should be the *first* that dies in the given time. Subtract this from the value of the sum (by Problem 9th) payable at the death of A, should that happen in the given time; and the remainder will be the answer.

EXAMPLE.

Suppose the age of A to be 30, that of B 25, and that of C 15. The term 10 years, and the proposed sum 1000*l*.—The first value (by Problem 31st) may be found equal to 113.23; and the second value (by Problem 9th and the *Northampton Table*) equal to 130.115. Their difference therefore,

fore, or 16.885 *l*, is the value required in *one present* payment.

PROBLEM XXXIX.

To find the value of a given sum payable at the death of C, provided A should be the first, B the second, and C the third that fails of the three lives A, B, and C, in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find by Problem 33d the value of the given sum payable at the death of C, should his life be the last that fails of the three lives in the given time; and *half* this value will be the answer.

EXAMPLE.

Let the age of A be 50, that of B 40, and that of C 30. The term 16 years, and the proposed sum 1000 *l*.—By Problem 33d the value of the given sum at the death of C, should his life be the last that fails in the given time is 9 *l*.—Consequently 4.5 *l*. is the value required in the present case in *one* payment.

PROB-

PROBLEM XL.

To find the value of an Annuity on the life of B *after* A, on the particular condition that A's life when it fails shall fail *after* the life of C, and also in a given term of years not exceeding the complement of the oldest life.

SOLUTION.

Find the complements of A and C for the given term as directed in Problem 30th.— Let D represent the life whose complement is the given term. Then find (by Problem 6th) the value (on the supposition of an equal decrement of life) of the three lives B, D, and D, taking double the expectation of B, according to the Table of Observations, for its complement *. Find also by the rule in page 55 the value of the longest of two equal lives whose complement is that of D. Multiply *half* the difference between those values into the square of the given term; and the product, divided by the complements of A and C for the given term

* In all the following Examples the *expectation* of life is taken from Table 4th, and the complements for the given term are computed from Table 3d, Appendix.

drawn into each other, will give the number of years purchase required.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50. The term 16 years.—In this case the age of D is 70.—By Problem 6th and under the restrictions mentioned above, the value of an Annuity on the longest of three lives aged 70, 30, and 70 is 15.724. By the rule in page 55 the value of the longest of two equal lives aged 70 is 8.026: *half* the difference, therefore, between these two values is 3.589, which being multiplied into 256 (the square of 16) and then divided by 1585.35 (the product of the complements of A and C for the given term into each other) gives .579 for the number of years purchase required.

COROLLARY.

If the Annuity *after* the Survivorship mentioned in this Problem, is to be continued on the life of B only for the *remainder of the given term*, instead of the *whole duration of the said life*; the value will be determined as follows :

“ From

“ From the value of an Annuity certain
 “ for the given term (found by Table 2d)
 “ subtract the value of an Annuity on the
 “ longest of two equal lives of the age of
 “ D (found by the rule in page 55) and
 “ *reserve* the remainder.—Multiply the dif-
 “ ference, between the value of an Annuity
 “ certain for the given term and the value of
 “ an Annuity on the longest of three equal
 “ lives of the age of D, (found by Problem
 “ 6th) into the given term. Divide the pro-
 “ duct by the complement of B for the given
 “ term, and take *half* the difference between
 “ this quotient and the *reserved* remainder.
 “ —Then say, as the complements of A and
 “ C for the given term drawn into each
 “ other are to this *half* difference, so is the
 “ square of the given term to a fourth pro-
 “ portional, which will be the answer.”

EXAMPLE.

Let the ages of A, B, and C be respec-
 tively 40, 30, and 50.—And let the term
 also be 16 years. The value of an Annuity
 certain for 16 years is 11.652. The value
 of the longest of two equal lives aged 70 is
 8.026. The remainder to be *reserved*, there-
 fore, is 3.626.—The value of the longest
 of three equal lives aged 70 is 8.971, which
 sub-

subtracted from 11.652 leaves 2.681. This difference being multiplied into 16 (the given term) and then divided by 58.98 (the complement of B for the given term) quotes .727; which again subtracted from 3.626, *reserved* above, and the remainder divided by 2, we have 1.449.—Hence as 1585.35 (the complements for the given term of A and C drawn into each other) is to 1.449 so is 256 (the square of 16) to .234, the number of years purchase required.

PROBLEM XLI.

To find the value of an Annuity on the life of B after A, on the particular condition that A's life when it fails, shall fail *before* the life of C, and also in a given number of years not exceeding the complement of the oldest life.

SOLUTION.

Find by Problem 40th the value of an Annuity on the life of B after A, on condition that A's life shall fail *after* C within the given term.—Let D, as before, represent a life whose decrements are equal, and whose complement for the whole duration is the given term; then subtract the value

value of the two joint lives B and D (by Problem 3d) taking double the expectation of B by the Table of Observations for its complement, from the value of an Annuity on the single life of B. Multiply the remainder into the given term, and divide the product by the complement of A for the term (found by the rule in Problem 30th.) —The difference between this quotient and the value first found will be the number of years purchase required.

EXAMPLE.

Suppose the age of A to be 40, that of B 30, and that of C 50. The term 16 years. The first value, by Problem 40th, appears to be .579.—By Problem 3d the value of the joint lives D and B (the age of D being in this case 70) is 5.55, which subtracted from 14.79 (the value of the life of B by Problem 2d) leaves 9.24. This remainder being multiplied into 16 (the given term) and then divided by 45 (the complement of A for the given term) quotes 3.285. Consequently 2.706 is the number of years purchase required.

COROL-

COROLLARY.

When the Annuity is to be continued on the life of B only during *the remainder of the given term* after the Survivorship proposed in this Problem, instead of being continued during the *whole* of the said life, the value will be determined as follows :

“ From the value of an Annuity on the
 “ life of B for the given term (found by
 “ Problem 4th) subtract the value of an
 “ Annuity on the joint lives of A and B
 “ for the given term (found by Corollary
 “ 1st to Problem 4th) and *reserve* the re-
 “ mainder.—Find next by the Corollary
 “ to Problem 40th the value of an Annuity
 “ on the life of B after A on condition that
 “ the life of A shall fail *after* C within the
 “ given term; which value being subtracted
 “ from the *reserved* remainder will give the
 “ answer in this case.”

EXAMPLE.

The ages of A, B, and C being respectively 40, 30, and 50, let the term be 16 years.—The value of the joint lives of A and B for 16 years (by Corollary 1st, Problem 4th, and taking double the expectations
 of

of the lives by the *Northampton* Table for their complements) is 8.476; and the value of the single life of B for the same term (by Problem 4th) is 10.072. Therefore 1.596 is the remainder to be reserved.—By the Corollary to Problem 40th the second value appears to be .234; which being subtracted from 1.596, leaves 1.362 for the number of years purchase required.

PROBLEM XLII.

To find the present worth of an estate or Annuity to be entered upon at the decease of A and B, provided both of them shall die in a given term not exceeding the complement of the oldest of the three lives, and that C shall survive one of them in particular (A).

SOLUTION.

Find the complements of A, B, and C for the given term as directed in Problem 30th, and let D, as in the two preceding Problems, represent a life whose complement for the whole duration is the given term, and whose decrements are equal.—Then find by the rule in page 55 the value of an Annuity on the longest of two equal lives of the age
of

of D, which value being subtracted from the perpetuity, multiply the remainder into the square of the given term; and dividing the product by the complements of A and B for the given term drawn into each other, let the quotient be *reserved*.

Find next by Problem 6th the value of an Annuity on the longest of three equal lives of the age of D; subtract this value from the perpetuity, and take *half* the remainder. Then say; as the complements of A, B, and C for the given term multiplied into each other, are to this *half* remainder, so is the cube of the given term to a fourth proportional; which being subtracted from the *reserved* quotient will give the answer*.

EXAMPLE.

Let the age of A be 40, that of B 30, and that of C 50. The term 16 years.—Here the age of D being 70, the value of an Annuity on the longest of two such equal

* In this and the two preceding Problems it may be observed that the Solutions are confined to the values of reversionary *estates* or *annuities*. But the values of reversionary *sums* are very easily obtained from hence, only “ by multiplying the number of years purchase into the “ given sum, and then dividing the product by the per- “ petuity increased by unity.”

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lives is 8.026.—The difference between this value and 25 (the perpetuity) being multiplied into 256 (the square of 16) becomes equal to 4345.344; and this product divided by 2614 (the complements of A and B for the given term drawn into each other) gives 1.662 for the quotient to be *reserved*.—The value of the longest of three equal lives aged 70 is 8.971, and *half* the difference between this value and 25 (the perpetuity) is 8.014. Hence as 93504 (the product of the complements of A, B, and C for the given term) is to 8.014 so is 4096 (the cube of 16) to .351. And .351 subtracted from 1.662 (the *reserved* quotient) leaves 1.311 for the number of years purchase required.



A P P E N D I X.

Note (A). See Chap. 2d, Page 57.

LET r be 1 $l.$ increased by its interest for a year, b the number of the living in the Table opposite to the present age of the given life, and $c, d, e, \&c.$ the numbers of the living at the end of 1, 2, 3, $\&c.$ years from his present age. Then, from what has been said in page 43, the value of an Annuity on this life will be $\frac{c}{br} + \frac{d}{br^2} + \frac{e}{br^3} \&c. \&c.$

—Call this value N .—Let a signify the number of the living in the Table opposite to the age of a person one year younger than the given life, and it follows that the value of an Annuity on the life of this younger person will be expressed by

$\frac{b}{ar} + \frac{c}{ar^2} + \frac{d}{ar^3} + \frac{e}{ar^4}, \&c.$ —But this series is equal to $\frac{b}{ar} \times 1 + \frac{c}{br} + \frac{d}{br^2}, \&c. = \frac{b}{ar} \times$
 $\frac{1}{1 + N}$ agreeable to the theorem in page 57.

Note (B). See Chap. 2d, Page 58.

In the case of the *youngest* life, it is evident there is no other difference between the operations in this rule and the theorem in note (A), than that those in the former are begun at the last year and continued upwards to the first year of life, and that those in the latter are begun at the first year and continued downwards to the last year of life. Let $a, b, c, d, e, \dots z$ express the numbers of persons at all ages in the Table, n the difference between the ages of the youngest and oldest life, and r 1 *l.* increased by its interest for a year.—

Then will the value of the youngest life by the theorem be $\frac{b}{ar} + \frac{c}{ar^2} + \frac{d}{ar^3} + \frac{e}{ar^4}, \&c. \dots +$

$\frac{z}{ar^n}$, and by the *Rule* the sum of all the values of

1 *l.* payable if this life exists to the several ages of the oldest, 2d, 3d, 4th, &c. lives is $\frac{z}{ar^n} + \frac{y}{ar^{n-1}}$

$+ \frac{x}{ar^{n-2}} + \frac{u}{ar^{n-3}} \dots + \frac{b}{ar}$.—The value

of an Annuity found by the *Theorem* is, in the present instance, to be multiplied by unity, and will, therefore, remain unaltered, and be just the same with the value found by the *Rule*.

Again; the value of a life one year older by the

Theorem is $\frac{c}{br} + \frac{d}{br^2} + \frac{e}{br^3} \dots + \frac{z}{br^{n-1}}$,

which being multiplied by the value of 1 *l.* payable if

if the youngest life should continue one year, or by $\frac{b}{ar}$, is equal to $\frac{c}{ar^2} + \frac{d}{ar^3} + \frac{e}{ar^4} + \dots + \frac{z}{ar^n}$. Now the sum of the values of 1 *l.* payable if the youngest life should exist to the age of the oldest and all the intermediate ages between it and a life one year older than the youngest, is expressed by the same fraction in a contrary order, or by $\frac{z}{ar^n} + \frac{y}{ar^{n-1}} + \frac{x}{ar^{n-2}} + \dots + \frac{c}{ar^2}$.—The value of a life two years older by the *Theorem* is $\frac{d}{cr} + \frac{e}{cr^2} + \frac{f}{cr^3} + \dots + \frac{z}{cr^{n-2}}$, which being multiplied by 1 *l.* payable if the youngest life exists two years, or by $\frac{c}{ar^2}$, is equal to $\frac{d}{ar^3} + \frac{e}{ar^4} + \frac{f}{ar^5} + \dots + \frac{z}{ar^n}$; and this is easily determined to be the sum of all the values of 1 *l.* payable if the youngest life exists to the age of the oldest and all the intermediate ages between it and a life two years older than the youngest.—Lastly, the value of an Annuity on the oldest life but one is equal to $\frac{z}{yr}$, which being multiplied by the value of 1 *l.* payable if the youngest life exists to this age, or by $\frac{y}{ar^{n-1}}$, is equal to $\frac{z}{ar^n}$; and this is evidently the value of 1 *l.* payable if the youngest life exists to the age of the oldest.—From hence the truth of the rule is manifest.

Note (C). PROBLEM X. Page 117.

The truth of this solution will appear from the following demonstration. Suppose the amount required of 56 or (n) Annuities, each 1 *l*, on 56 (or n) lives all aged 30.—In this case, as one life will drop every year by *De Moivre's* hypothesis†, the amount of the several Annuities will be as follows;

Annuity

1. ---- 0

1. ---- 1

1. ---- 1 + r^* .1. ---- 1 + r + r^2 .1. ---- 1 + r + r^2 + r^3 .&c. &c. &c. &c. &c. to n terms.

Sum 56 -- $n-1$. + $n-2$. r + $n-3$. r^2 + $n-4$. r^3 , &c.
 = the amount of all the Annuities, or of n Annuities on n lives aged 30.—Therefore the amount of one Annuity of 1 *l*. on one life aged 30 must be $\frac{n-1}{n}$ + $\frac{n-2}{n}$. r + $\frac{n-3}{n}$. r^2 , &c. divided by n , or $\frac{n-1}{n} + \frac{n-2}{n} \times r + \frac{n-3}{n} \times r^2$, &c. to n terms.—In order to sum up this series it may be resolved into the two following ones; $1 + r + r^2 + r^3 + \dots + r^{n-1}$ and $-\frac{1}{n} \times 1 + 2r + 3r^2 + 4r^3 + \dots + nr^{n-1}$. The first of these is easily found equal to $\frac{r^n - 1}{r - 1}$, and

† See this hypothesis explained in a note, page 3.

* r denotes 1 *l*. increased by its interest for a year.

the second equal to $\frac{r^n}{r-1} - \frac{r^n-1}{n \cdot r-1}$. Consequently the whole amount of the Annuity will be expressed by $\frac{r^n-1}{n \cdot r-1} - \frac{1}{r-1}$, which is the general rule.

Dr. Price in the Philosophical Transactions (Vol. 66, No. 6.) has given the different methods of determining the values of Annuities certain, and Life-Annuities payable *yearly*, *half-yearly*, *quarterly*, and *momently*.—By reasoning much in the same manner as he has done, the following theorems may be obtained for determining the *amounts* of Annuities laid up *yearly*, *half-yearly*, &c. either during a given number of years, or during the continuance of a given life.

Let r be the interest of 1 *l.* for a year, P the perpetuity, n the term of years during which the *annuity certain* is to accumulate, y the sum to which this Annuity will amount when laid up *yearly*, b the same sum when it is laid up *half yearly*, q the same sum when it is laid up *quarterly*, and m the same sum when it is laid up *momently*.—Let also Y , H , Q , and M denote the respective sums to which the Annuity will amount when laid up yearly, half-yearly, quarterly, or *momently* during the continuance of a life whose complement is n .—Then

$$\text{THEOREM I.} \dots y = \frac{1+r}{r} - P$$

$$\text{THEOREM II.} \dots b = \frac{1 + \frac{r}{2}}{r} - P$$

$$\text{THEOREM III.} \dots q = \frac{1 + \frac{r}{4} \sqrt[4n]{}}{r} - P$$

$$\text{THEOREM IV.} \dots m = \frac{N}{r} - P, \text{ where } N \text{ stands for the number which has } rn \text{ for its hyperbolic logarithm,}$$

$$\text{THEOREM V.} \dots Y = \frac{y}{nr} - P$$

$$\text{THEOREM VI.} \dots H = \frac{b}{nr} - P$$

$$\text{THEOREM VII.} \dots Q = \frac{q}{nr} - P$$

$$\text{THEOREM VIII.} \dots M = \frac{m}{nr} - P$$

If n , or the term of years, be 50, and r , or the interest of 1 $l.$ for a year, be .05, then the amount of an *Annuity certain* by Theorem 1st will be = 209.34 $l.$, by Theorem 2d it will be = 216.28 $l.$, by Theorem 3d it will be = 219.84 $l.$, and by Theorem 4th it will be = 224 $l.$

Again; if n be the complement of a life whose age is 36, which by *De Moivre's* hypothesis is 50, and r , as before, be .05, then the amount of 1 $l.$ *per annum* during this life will, by Theorem 5th, be = 63.73 $l.$, by Theorem 6th = 66.49 $l.$, by Theorem 7th = 67.93 $l.$, and by Theorem 8th = 69.6 $l.$

Note

Note (D). PROBLEM XI. Page 119.

Let V be the value of a life whose complement is n , v the value of 1 $l.$ at the end of the first year, 2 $l.$ at the end of the second year, &c.; A the value of an Annuity certain for $n-1$ years; G the value of two equal joint lives whose complement is n ; and r , as usual, 1 $l.$ increased by its interest for a year. Then by reasoning as in page 43 and sup-

posing an equal decrement of life, $v = \frac{n-1}{nr} +$

$$\frac{2 \cdot \overline{n-2}}{nr^2} + \frac{3 \cdot \overline{n-3}}{nr^3}, \&c. = n \times \frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3}$$

$$\&c. = n \times \frac{1^2}{n \cdot nr} + \frac{2^2}{n \cdot nr^2} + \frac{3^2}{n \cdot nr^3}, \&c. \text{ But}$$

$$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3}, \&c. =$$

$$V. \text{ Therefore } n \times \overline{A-V} = n \times \frac{1}{nr} + \frac{2}{nr^2} + \frac{3}{nr^3},$$

$$\&c. \text{ --- Also } \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{2}{nr} + \frac{4}{nr^2} + \frac{6}{nr^3},$$

$$\&c. + \frac{1^2}{n \cdot nr} + \frac{2^2}{n \cdot nr^2} + \frac{3^2}{n \cdot nr^3}, \&c. = G \text{ ---}$$

But the first of these three serieses is equal A , and the second $= 2 \times \overline{A-V}$: Therefore the third, or

$$\frac{1^2}{n \cdot nr} + \frac{2^2}{n \cdot nr^2} + \frac{3^2}{n \cdot nr^3}, \&c. = G + 2 \times \overline{A-V}$$

$$= A; \text{ hence } n \times \frac{1^2}{n \cdot nr} + \frac{2^2}{n \cdot nr^2} + \frac{3^2}{n \cdot nr^3}, \&c. = n$$

$$\times \overline{G-A}$$

$\times \overline{G-A} + 2n \times \overline{A-V}$, which being subtracted from $n \times \overline{A-V}$ found above we shall at last have $v = n \times \overline{V-G} \dots Q. E. D.$

This rule may be also very easily deduced from Dr. Price's Theorem, given in his Treatise on Reversionary Payments, page 302, 3d edition.—The same symbols being retained, let P denote the perpetuity, and p the value of 1 $l.$ at the end of n years; then will Dr. Price's Theorem for the whole continuance of life be expressed by $\overline{A-G} \times n + n.p.P - A.P.r = n \times \overline{A} + \overline{Pp} - APr - nG \dots$
But $A + Pp$ is $= P$, and $P - \frac{APr}{n}$ is known to be $= V$. Therefore $n \times \overline{A} + \overline{Pp} - APr = nV$; and $n \times \overline{A} + \overline{Pp} - APr - nG = n \times \overline{V-G}$.

Note (E). PROBLEM XII. Page 128.

Let the value of the two *single* lives be denoted by A and B ; the value of the two *joint* lives by D ; and the given sum by S .—Then since the *whole* Annuity is to be enjoyed till the extinction of the joint lives, the value of an Annuity of 1 $l.$ till this period will be expressed by D .

The value of one life after another is known to be equal to the *difference* between the *joint* lives and the *life in expectation*. Therefore the value of *half* the Annuity on the supposition that A will be the survivor is $\frac{A-D}{2}$; and the same value on the

sup-

supposition that B will be the survivor, is $\frac{B-D}{2}$

---- The sum of those two values, or $\frac{A+B}{2} - D$, being added to D (the value of the whole Annuity during the joint lives) we have $\frac{A+B}{2}$ for the number of years purchase such an Annuity is worth during the continuance of the joint lives, and the remainder of the surviving life. Hence as $\frac{A+B}{2}$ is to an Annuity of 1 *l.* (under the restrictions mentioned in the Problem) so is S to $\frac{2S}{A+B}$, which is the general rule.

Note (F). PROBLEM XXIII. Page 154.

CASE 1st. Where B is supposed to be the oldest life.

—Let *a*, *b* and *c*, denote the complements of the lives of A, B and C, and let A be the youngest. The values of the 1st, 2^d, 3^d, &c. rents to *b* rents will be the probability, that all the three lives will have failed at the beginning of the 2^d, 3^d, 4th, &c. years to *b* years, (the life of *c* having failed first) multiplied respectively by $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c. to $\frac{1}{r^b}$

(*r* denoting 1 *l.* increased by its interest for a year)

—The probability that any single life will have failed in any number of years is that number divided by the complement of life. The probabilities,

ties, therefore, that all the three lives whose complements are a , b and c will have failed in 1, 2, 3, &c. years (that is, at the beginning of the 2d, 3d, 4th, &c. years to b years) are $\frac{1^3}{abc}$, $\frac{2^3}{abc}$, $\frac{3^3}{abc}$, &c. to b terms. The probabilities that they shall have all failed at the beginning of the 2d, 3d, 4th, &c. years to b years, and that any one in particular of them (suppose the life whose complement is c) shall have failed first, are $\frac{1^3}{3abc}$, $\frac{2^3}{3abc}$, $\frac{3^3}{3abc}$, &c. to $\frac{b^3}{3abc}$ ($= \frac{b^2}{3ac}$). The probability, therefore, that they shall have all failed (C having failed first) in b years, or at the beginning of the $\overline{b+1}$ th year, is $\frac{b^2}{3ac}$; from whence it follows that the present value of the 1st, 2d, 3d, &c. rents to b rents is $\frac{1^3}{3abcr} + \frac{2^3}{3abcr^2} + \frac{3^3}{3abcr^3}$, &c. to $\frac{b^2}{3acr^b} = \frac{b^2}{3ac} \times \frac{1^3}{b^3r} + \frac{2^3}{b^3r^2} + \frac{3^3}{b^3r^3}$, &c. to b terms; and that the value of the rents ever after, arising from the probability that all the lives shall die in b years (C having died first) is $\frac{b^2}{3acr^b}$ multiplied into $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, &c. *ad infinitum*.

—But there is a further value of the several rents after b years, arising from the probability of A's dying *after* the expiration of this time, restrained how-

however to the contingency of C's having been survived by B. The probabilities that A dies after the expiration of b , $\overline{b+1}$, $\overline{b+2}$, &c. years are $\frac{1}{a}$, $\frac{2}{a}$, $\frac{3}{a}$, &c. The probability that C, the youngest of the two lives B and C, will die before B in b years is $\frac{b^2}{2bc} = \frac{b}{2c}$; which therefore multiplied into $\frac{1}{ar^{b+1}}$ will give the additional value of the $\overline{b+1}$ th rent, and will be equal to $\frac{b}{2cr^b} \times \frac{1}{ar}$. In the same manner the additional values of the $\overline{b+2}$ d, $\overline{b+3}$ d, &c. rents will be $\frac{b}{2cr^b} \times \frac{2}{ar^2}$, $\frac{b}{2cr^b} \times \frac{3}{ar^3}$, &c. The sum of all these values to the extremity of A's life, or to $\overline{a-b}$ terms, will be $= \frac{b}{2cr^b} \times \frac{1}{ar} + \frac{2}{ar^2} + \frac{3}{ar^3}$, ----- &c. to $\frac{a-b}{ar^a}$. And the value of these rents ever after, arising from the probability of A's dying after B, and C's being survived by B, is $\frac{b}{2cr^a} \times \frac{a-b}{a} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, *ad infinitum*, $= \frac{b}{2cr^a} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, &c. $-\frac{bb}{2acr^a} \times \frac{1}{r} + \frac{1}{r^2}$, &c. The whole present value, therefore, of all the rents is $\frac{b^2}{3ac} \times \frac{1^3}{b^3r} + \frac{2^3}{b^3r^2} + \frac{3^3}{b^3r^3}$ ----- $(b) +$

$$\begin{aligned}
& \frac{b^2}{3acr^2} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \text{ \&c. ad infinitum, } + \frac{b}{2cr^2} \\
& \times \frac{1}{ar} + \frac{2}{ar^2} + \frac{3}{ar^3} \text{ ----- } \overline{(a-b)} + \frac{b}{2cr^2} \times \\
& \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \text{ \&c. ad infinitum, } - \frac{bb}{2acr^2} \times \\
& \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \text{ \&c. ad infinitum. } \text{ Let } P = \text{ the} \\
& \text{value of the perpetuity, } B = \text{ the value of the} \\
& \text{longest of three equal lives whose complement is} \\
& b, A = \text{ the value of a single life, whose comple-} \\
& \text{ment is } a, \text{ for } \overline{a-b} \text{ years. Then the foregoing ex-} \\
& \text{pression will be } = \frac{b^2}{ac} \times \frac{P-B}{3} - \frac{P}{2r^2} + \frac{b}{2cr^2} \times \\
& P-A, \text{ which is the general rule.}
\end{aligned}$$

The solution of the second Case is so very easy as hardly to require an explanation. For since the absolute value of the Reversion is composed of three parts, depending on the chance of B's dying before A and C, of A's dying before B and C, and C's dying before A and B, it is evident, that if the sum of the two former be subtracted from the *whole* value, the remainder will give the number of years purchase required.

Note (G). PROBLEM XXVII. Page 166.

CASE FIRST. *When B is oldest, or when B is older than C, and C is older than A.* Let a, b, c and r denote the same quantities as in the preceding demon-

demonstration. The values of the 1st, 2d, 3d, &c. rents to b rents may be considered as depending upon either of two events. First, that A and C have *both* failed before the beginning of the 2d, 3d, &c. years, C having failed last, and B being alive at those periods; secondly, that B and C are both alive and A dead at the beginning of the 2d, 3d, &c. years. The first of these probabilities is

$$\frac{\overline{b-1.1^2}}{2abc} \text{ for the 1st year, } \frac{\overline{b-2.2^2}}{2abc} \text{ for the 2d}$$

year, and so on to b years. The last of these probabilities for the 1st, 2d, 3d, &c. years will be

$$\frac{\overline{b-1.c-1.1}}{abc}, \frac{\overline{b-2.c-2.2}}{abc}, \&c. \text{ to } b \text{ terms.}$$

Therefore the value of the 1st, 2d, 3d, &c. rents on

$$\text{the first contingency is } \frac{\overline{b-1.1^2}}{2abcr} + \frac{\overline{b-2.2^2}}{2abcr^2} \dots$$

--- (b) and the value of the same rents on the last

$$\text{contingency is } \frac{\overline{b-1.c-1.1}}{abcr} + \frac{\overline{b-2.c-2.2}}{abcr^2}$$

+ &c. ----- (b). Let BC = the value of the

joint lives B and C, ABC = the value of the

three joint lives, $\overset{1}{B} \overset{1}{C}$ = the value of the longest

of the two lives B and C, and $\overset{1}{A} \overset{1}{B} \overset{1}{C}$ = the value

of the longest of the three lives. Then will

the last of the above serieses be expressed by

$\overline{BC - ABC}$, and the first of them by

$$\frac{\overset{1}{A} \overset{1}{B} \overset{1}{C} - \overset{1}{A} \overset{1}{C}}{2}; \text{ the sum of which is the general}$$

rule.

CASE

CASE SECOND. *When B is younger than C, and C older than A.* By reasoning as above, the value of the 1st, 2d, 3d, &c. rents to c rents will be

$$\frac{\overline{b-1.} \cdot \overline{c-1.} \cdot 1}{abcr} + \frac{\overline{b-2.} \cdot \overline{c-2.} \cdot 2}{abcr^2} + \&c. \dots (c)$$

$$\text{and } \frac{\overline{b-1.} \cdot 1^2}{2 abcr} + \frac{\overline{b-2.} \cdot 2^2}{2 abcr^2} \dots (c). \text{ Now it}$$

is evident that B has a further chance of receiving the Annuity after this period depending upon the probability that he has survived both A and C, A having died first. The value therefore of the $c+1$ th, $c+2$ d, &c. rents to the extremity of B's life will be this probability multiplied into

$$\frac{1}{r^{c+1}}, \frac{1}{r^{c+2}}, \&c. (b-c) \text{ that is, it will be } =$$

$$\frac{\overline{b-c-1.} \times cc}{2 abcr^{c+1}} + \frac{\overline{b-c-2.} \times cc}{2 abcr^{c+2}} + \&c. \dots$$

$(b-c)$. Hence the whole present value of the

$$\text{Annuity will be } \frac{\overline{b-1.} \cdot \overline{c-1.} \cdot 1}{abcr} + \frac{\overline{b-2.} \cdot \overline{c-2.} \cdot 2}{abcr^2}$$

$$\dots (c) + \frac{\overline{b-1.} \cdot 1^2}{2 abcr} + \frac{\overline{b-2.} \cdot 2^2}{2 abcr^2} + \dots$$

$$\frac{\overline{b-c.} \cdot c^2}{2 abcr^c} + \frac{\overline{b-c-1.} \cdot c^2}{2 abcr^{c+1}} + \dots (b-c)$$

With regard to the last of these serieses; since

$$\frac{1}{ccb} \times \frac{\overline{b-1.} \cdot 1^2}{r} + \frac{\overline{b-2.} \cdot 2^2}{r^2} \dots + \frac{\overline{b-c.} \cdot cc}{r^c} +$$

$$\frac{\overline{b-c-1.} \cdot cc}{r^{c+1}} \dots (b-c) \text{ is the value of the Re-}$$

version

version of the life of B after the longest of two equal lives C, C; it follows that the true value thereof must be equal to that of the said Reversion multiplied by $\frac{c}{2a}$. Let the same symbols be re-

tained as in the preceding case, and let $\overset{1}{C} \overset{1}{C} =$ the value of the longest of two equal lives C, C, and $\overset{1}{B} \overset{1}{C} \overset{1}{C} =$ the value of the longest of three lives B, C and C; then will the whole value of the Annuity in this case be expressed by $\overset{1}{B} \overset{1}{C} - \overset{1}{A} \overset{1}{B} \overset{1}{C} + \frac{c}{2a} \times \overset{1}{B} \overset{1}{C} \overset{1}{C} - \overset{1}{C} \overset{1}{C}$.

CASE THIRD. *When B is younger than C, and C younger than A.* The truth of this rule is very obvious: For since the *absolute* value of the Reversion of the life of B after the joint lives of A and C is composed of two parts, depending on the chance of the younger dying before the elder, and on the chance of the elder dying before the younger, it follows, that if the value of the expectation on the *first* of these contingencies (given in the solution of Case 2d) be subtracted from the *absolute* value of the Reversion, that the remainder will be the value of the expectation on the *latter* of them.

Note (H). PROBLEM XXIX. Page 175.

CASE FIRST. *When C is oldest.* Let a, b, c and r still denote the same quantities: Then since it appears from the demonstration of Problem 23d

Q

that

that $\frac{1^3}{3abc}$, $\frac{2^3}{3abc}$, &c. represent the probabilities that the three lives shall have failed at the beginning of the 2d, 3d, &c. years, and that one of them in particular (suppose the life whose complement is a) shall have failed first; it is evident that if the foregoing fractions are severally diminished one half, they will express the probability of survivorship required in this Problem; that is, $\frac{1^3}{3abc}$

$\times \frac{1}{2}$, $\frac{2^3}{3abc} \times \frac{1}{2}$, &c. will denote the probabilities that the three lives are dead at the beginning of the 2d, 3d, &c. years, and that A has died first, and either of the other two (C for instance) has died last. For as it is an equal chance which of the two surviving lives B and C shall die first, it is plain that if the former probabilities *without* this restriction are lessened in the proportion of $\frac{1}{2}$,

we shall have the probabilities of survivorship *with* this restriction.—The value, therefore, of the rents during the 1st, 2d, 3d, &c. years to c years will be

$$\frac{1^3}{6abr} + \frac{2^3}{6abr^2} + \text{\&c.} \text{-----} (c) = \frac{c^3}{6ab} \times$$

$$\frac{1^3}{c^3r} + \frac{2^3}{c^3r^2} + \text{-----} \frac{c^3}{c^3r^c}, \text{ and the values of all}$$

$$\text{the rents ever after will be } \frac{c^3}{6abr} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$$

ad infinitum; for $\frac{c^3}{6abc}$ expresses the probability of

survi-

survivorship during the whole continuance of C's life, at the extinction of which A and B must be dead (B having died last) to entitle to the receipt of the following rents; that is, to the rents after c

years. Now those rents being simply $= \frac{1}{r^{c+1}} + \frac{1}{r^{c+2}} + \frac{1}{r^{c+3}} \&c. \text{ ad infinitum}$, if they are diminished in the proportion of the probability of this survivorship, or multiplied by $\frac{c^2}{6ab}$, we shall have

their true value. But $\frac{c^2}{6ab} \times \frac{1}{r^{c+1}} + \frac{1}{r^{c+2}} \&c.$

is the value of the perpetuity after c years multiplied into $\frac{c^2}{6ab}$, and the first series is equal to the difference between an Annuity certain for c years and the value of the longest of three equal lives, whose complement is c , multiplied into the same co-efficient: Hence if P = the perpetuity, and C = the value of the three lives just mentioned, the

two serieses $\frac{c^2}{6ab} \times \frac{1^3}{c^3 r} + \frac{2^3}{c^3 r^2} + \dots + \frac{c^3}{c^3 r^c}$,

and $\frac{c^2}{6abr^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. \text{ ad infinitum}$,

expressing the whole present value of the Reversion, will become equal to $\frac{c^2}{6ab} \times \overline{P-C}$, which is the general rule.

CASE SECOND. *When B is oldest.* By reasoning as in the preceding Case, the value of the rents for the 1st, 2d, 3d, &c. years to b years are

$$\frac{1^3}{6abc}, \frac{2^3}{6abc^2}, \frac{3^3}{6abc^3}, \text{ \&c. to } \frac{b^3}{6abc^b} = \frac{b^2}{6ac}$$

$$\times \frac{1^3}{br^3} + \frac{2^3}{b^2r^2}, \text{ \&c. to } \frac{b^3}{b^3r^b}, \text{ and the value of}$$

the rents ever after, that is of the $\overline{b+1}$ th, $\overline{b+2}$ d, &c. rents, *ad infinitum*, arising from the probability that all the three lives shall die in b years, A

having died first and C having died last, is $\frac{b^2}{6ac} \times$

$$\frac{1}{r^{b+1}} + \frac{1}{r^{b+2}} + \frac{1}{r^{b+3}}, \text{ \&c. } = \frac{b^2}{6acr^b} \times$$

$$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \text{ \&c. ad infinitum. But there is a}$$

further expectation beyond this period (or the limit of B's life) depending on the probability of C's dying *after* B and A's dying *before* him. The first of these probabilities for the 1st, 2d, 3d, &c.

years to $\overline{c-b}$ years, is $\frac{1}{c}, \frac{2}{c}, \frac{3}{c}, \text{ \&c. to } \overline{c-b}$

terms. The last is constantly expressed by $\frac{b}{2a}$.

Hence the additional value of the rents after the limit of B's life for the remainder of C's existence,

will be for the $\overline{b+1}$ th year $\frac{1}{cr^{b+1}} \times \frac{b}{2a}$, for the

$\overline{b+2}$ d year $\frac{2}{cr^{b+2}} \times \frac{b}{2a}$, and so on to the $\overline{c-b}$ th

year,

year, the value of which rent will be $\frac{c-b \times b}{2car^c}$.

These several values being added together, will be

$$\text{equal } \frac{b}{2ar^b} \times \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \dots + \frac{c-b}{cr^{c-b}}.$$

Again, the values of the rents for ever after the limit of C's life, arising from the probability that C shall have died *after* B, and that B shall have survived A, will be for the $c+1$ th rent $\frac{c-b \cdot b}{2ca} \times$

$\frac{1}{r^{c+1}}$, for the $c+2$ d, $c+3$ d, &c. rents, *ad infinitum*,

$$\frac{c-b \cdot b}{2ca} \times \frac{1}{r^{c+2}}, \frac{c-b \cdot b}{2ca} \times \frac{1}{r^{c+3}}, \&c.$$

$$= \frac{c-b \cdot b}{2acr^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{b}{2ar^c} \times$$

$$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. - \frac{bb}{2acr^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3},$$

&c. The whole present value, therefore, of all

the rents is $\frac{b^2}{6ac} \times \frac{1^3}{b^3r} + \frac{2^3}{b^3r^2} + \frac{3^3}{b^3r^3} \dots (b)$

$$+ \frac{b^2}{6acr^b} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. - \frac{bb}{2acr^c} \times$$

$$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. + \frac{b}{2ar^b} \times \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3}$$

$$\dots - \frac{c-b}{2ar^c} + \frac{b}{2ar^c} \times \frac{1}{r} + \frac{1}{r^2}, \&c. \text{ Let P}$$

= the value of the perpetuity, B = the value of the longest of three equal lives whose complement

is b , and β = the value of an Annuity on the life of C for $c-b$ years, then the foregoing expression

$$\text{will be } \frac{b^2}{ac} \times \frac{P-B}{6} - \frac{P}{2r} + \frac{b}{ar} \times \frac{P-\beta}{2}$$

----- which is the rule.

CASE THIRD. *When A is oldest and B youngest.*

By proceeding as in the two former Cases, it will appear that the value of all the rents during the limit of A's life, that is for the 1st, 2d, 3d, &c.

$$\text{years to } a \text{ years, will be } \frac{a^2}{6bc} \times \frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} +$$

$$\frac{3^3}{a^3r^3} \text{ ----- } (a), \text{ and that the value of all the rents}$$

ever after, arising from the probability that the three lives shall die in a years, B having died 2d,

$$\text{and C having died last, will be } \frac{a^2}{6bcr^2} \times \frac{1}{r} + \frac{1}{r^2}$$

&c. But this last series by no means expresses the whole value of the $a+1$ th, $a+2$ d, &c. rents.

There are two other contingencies, upon the happening of either of which the expectant may come into possession of the estate. First, that B and C

are living at the end of a years, and that they shall

both be dead at the beginning of the $a+2$ d, $a+3$ d,

&c. years, (C having previously survived B); the

probabilities of which contingency are severally

$$\text{expressed by } \frac{1^2}{2bc}, \frac{2^2}{2bc}, \frac{3^2}{2bc}, \text{ \&c. The value,}$$

therefore, of the $a+1$ th rent arising from the

$$\text{chance of this event's happening is } \frac{1^2}{2bcr^2+1}; \text{ the}$$

values

values of the $\overline{a+2d}$, $\overline{a+3d}$, &c. rents to $\overline{c-a}$ rents will be $\frac{2^2}{2 bcr^{a+2}}$, $\frac{3^2}{2 bcr^{a+3}}$, &c. to $\frac{\overline{c-a}^2}{2 bcr^c}$.

The sum of all which is equal to $\frac{c}{2 br^a} \times \frac{1^2}{c^2 r} +$

$$\frac{2^2}{c^2 r^2} + \frac{3^2}{c^2 r^3} + \dots + \frac{\overline{c-a}^2}{cc. r^{c-a}}.$$

And the value of all the rents ever after, arising from the probability that B and C are both dead at the end of c years, that C has died last, and that both of

them have survived the a first years, is $\frac{\overline{c-a}^2}{2 bcr^c} \times$

$$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \text{ ad infinitum.}$$

The second contingency, which may bring the expectant into possession of the estate, after the limit of A's life, arises from the probability of B's dying during the said limit (having previously survived A) and C's dying *after* this limit. The probability that B dies in a years after surviving A, is constantly expressed by $\frac{a}{2b}$. The probability that C is dead

at the beginning of the $\overline{a+1}$ th year is $\frac{1}{c}$, at the

beginning of the $\overline{a+2d}$ year $\frac{2}{c}$, and so on to the

$\overline{c-a}$ th year. The values therefore of the $\overline{a+1}$ th, $\overline{a+2d}$, &c. rents to $\overline{c-a}$ rents arising from the

probability of this contingency are $\frac{a}{2b} \times \frac{1}{cr^{a+1}}$,

$$\begin{aligned}
& \frac{a}{2b} \times \frac{2}{cr^{a+2}}, \&c. \text{ to } \frac{a}{2b} \times \frac{c-a}{cr^c} = \frac{a}{2br^a} \times \\
& \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \dots\dots\dots + \frac{c-a}{cr^{c-a}}. \text{ And the va-} \\
& \text{lues of the rents ever after, that is, the values of} \\
& \text{the } \overline{c+1}\text{th, } \overline{c+2}\text{d, } \overline{c+3}\text{d, } \&c. \text{ rents, } ad \text{ infinitum,} \\
& \text{arising from the probability that C has survived} \\
& \text{the first } a \text{ years, and that B has died during this} \\
& \text{time after A, are } \frac{a}{2b} \times \frac{c-a}{cr^c} \times \frac{1}{r}, \frac{a}{2b} \times \frac{c-a}{cr^c} \\
& \times \frac{1}{r^2}, \&c. = \frac{a}{2b} \times \frac{c-a}{cr^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \\
& \&c. ad \text{ infinitum. The whole present value, there-} \\
& \text{fore, of all the rents is } \frac{a^2}{6bc} \times \frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} + \\
& \frac{3^3}{a^3r^3} \dots\dots\dots (a)^3 + \frac{a^2}{6bcr^a} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. ad \\
& \text{infinitum, } + \frac{c}{2br^a} \times \frac{1^3}{c^3r} + \frac{2^3}{c^3r^2} + \frac{3^3}{c^3r^3} \&c. \\
& \dots\dots\dots (c-a) + \left(\frac{c-a}{2bcr^c} \right)^2 = \frac{c^2 - 2ac + a^2}{2bcr^c} \\
& \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} ad \text{ infinitum, } + \frac{a}{2br^a} \times \\
& \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \dots\dots\dots (c-a) + \frac{ac - a^2}{2bcr^c} \times \\
& \frac{1}{r} + \frac{1}{r^2} \&c. = \frac{a^2}{6bc} \times \frac{1^3}{a^3r} + \frac{2^3}{a^3r^2} + \frac{3^3}{a^3r^3} \dots\dots\dots \\
& \dots\dots\dots (a) + \frac{a^2}{6bcr^a} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. ad \text{ in-} \\
& \text{finitum,}
\end{aligned}$$

$$\begin{aligned} & \text{finitum,} + \frac{c}{2br^2} \times \frac{1^2}{c^2r} + \frac{2^2}{c^2r^2} \dots (\overline{c-a}) + \\ & \frac{c}{br^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. + \frac{a}{2br^a} \times \frac{1}{cr} + \\ & \frac{2}{cr^2} + \frac{3}{cr^3} \dots (\overline{c-a}) + \frac{a}{2br^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \\ & \&c. \text{ ad infinitum,} - \frac{a}{br^c} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \&c. \end{aligned}$$

ad infinitum ----- Let P = the value of the perpetuity, A = the value of the longest of three equal lives whose complement is a , K = the value of the longest of two equal lives, whose complement is c , for $\overline{c-a}$ years, and x = the value of a single life of the same age for the same term; then the foregoing expression will be = $\frac{a^2}{6bc} \times \overline{P-A} + \frac{c \cdot \overline{P-K} + a \cdot \overline{P-x}}{2bra} - \frac{aP}{br^c}$, which is the general rule.

CASE FOURTH. *When A is oldest and C youngest.*
By reasoning exactly as in Case 3d, the value of the rents during the limit of A and B's lives, and ever after, arising from the probability of all the three lives being dead at the end of b years, (A having died first and C having died last) will be found equal to $\frac{a^2}{6bc} \times \overline{P-A} + \frac{b \cdot \overline{P-B} + a \cdot \overline{P-\beta}}{2cr^a} - \frac{aP}{cr^b}$ (β and B representing the values of Annuities on the single and the longest of two equal lives,

lives, whose complement is b , for $\overline{b-a}$ years, and P and A representing the same values as in Case 3d).—Now it is evident that the above does not express the *whole* value of the estate: For the rents after b years; that is, after the necessary extinction of A and B 's lives, have an additional value from the probability of C 's surviving this period, and being dead at the beginning of the $\overline{b+2d}$, $\overline{b+3d}$, &c. years to $\overline{c-b}$ years. These probabilities are severally expressed by $\frac{1}{c}$, $\frac{2}{c}$, $\frac{3}{c}$, &c. ---- to $\frac{c-b}{c}$. Therefore the additional value of the $\overline{b+1th}$, $\overline{b+2d}$, $\overline{b+3d}$, &c. rents to the $\overline{c-bth}$ rent, will be $\frac{1}{cr^{b+1}} + \frac{2}{cr^{b+2}} + \frac{3}{cr^{b+3}} \text{ ---- to } \frac{c-b}{cr^c} =$
 $\frac{1}{r^b} \times \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \text{ ---- } \frac{c-b}{cr^{c-b}}$. And the value ever after, arising from the probability of C 's having survived b years, will be $\frac{c-b}{cr^c} \times \frac{1}{r} +$
 $\frac{c-b}{cr^c} \times \frac{1}{r^2}$ &c. *ad infinitum*, $= \frac{c-b}{cr^c} \times \frac{1}{r} +$
 $\frac{1}{r^2} + \frac{1}{r^3}$ &c. *ad infinitum*. Hence the whole additional value of the rents after b years will be $=$
 $\frac{1}{r^b} \times \frac{1}{cr} + \frac{2}{cr^2} + \frac{3}{cr^3} \text{ ---- } (c-b) + \frac{1}{r^c} \times$
 $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$ &c. $= \frac{b}{cr^c} \times \frac{1}{r} + \frac{1}{r^2}$ &c. $=$

$$\frac{\overline{P-C}}{r^c} - \frac{bP}{cr^c} \text{ (C representing the value of an An-}$$

nuity on a life, whose complement is c , for $c-b$ years). This expression, therefore, added to the above, will be the whole value of all the rents,

$$\text{and is equal to } \frac{a^2}{6bc} \times \overline{P-A} + \frac{b \cdot \overline{P-B} + a \cdot \overline{P-\beta}}{2cr^a}$$

$$- \frac{aP}{cr_b} + \frac{\overline{P-C}}{r^c} - \frac{bP}{cr^c}, \text{ which is the general rule.}$$

Mr. *Simpson* in his *Select Exercises* (page 328)

$$\text{has given the fluent } \frac{Sq^x}{2abcm} \times x^2 - \frac{2x}{m} + \frac{2}{m}$$

(when properly corrected) as the true value of the sum S to be received on these contingencies; q denoting unity divided by the rate of interest, m the hyperbolic logarithm of q , and x the interval of time during which the value is required. But this fluent by no means expresses the whole value, except in the single case where C is the oldest of the three lives.—By a *fluxional* calculus of the same kind with Mr. *Simpson's*, the following Theorems may be obtained, which express correctly the whole value in every case.

THEOREM

THEOR. 1st. $\left. \begin{array}{l} \\ \\ \text{When } C \text{ is} \\ \text{the oldest.} \end{array} \right\} = \frac{S}{2abc} \times \overline{A - pc} \times 2P^2 - Ppc^2$

----- A being the value of an Annuity certain for the time c , p the value of 1 $l.$ at the end of c years, r the value of 1 $l.$ increased by its interest for a year, P the perpetuity, S the given sum, and a, b, c the respective complements of A, B , and C .

THEOR. 2d. $\left. \begin{array}{l} \\ \\ \text{When } B \text{ is} \\ \text{oldest.} \end{array} \right\} = \frac{S}{2abc} \times \overline{B - \pi b} \times 2P^2 - P\pi b^2 +$

$\frac{Sb}{2ac} \times \frac{N}{r^{b+1}}$ ----- S, a, b, c, r , and P being as in Theorem 1st, B the value of an Annuity certain for b years, π the value of 1 $l.$ at the end of b years, and N the value of an Annuity certain for $c - b$ years.

THEOR. 3d. $\left. \begin{array}{l} \\ \\ \text{When } A \text{ is} \\ \text{oldest and} \\ B \text{ youngest.} \end{array} \right\} = \frac{S}{2abc} \times \overline{C - \chi a} \times 2P^2 - P\chi a^2 +$

$\frac{S}{bc} \times \frac{\overline{A - a} \times \overline{c - a}}{r^{a+1}} + \frac{Aa}{2r^{a+1}}$ ----- S, a, b, c, r , and P being as above, C the value of an Annuity certain for (a) years, χ the value of 1 $l.$ at the end of (a) years, A the value of an Annuity certain for $c - a$ years, and a the value of an Annuity on the life whose complement is $c - a$.

THEOR. 4th. }
$$= \frac{S}{2 abcr} \times \overline{C - xa} \times 2P^a - Pxa^2$$

When *A* is }
$$+ \frac{S}{c} \times \frac{A - \beta}{r^{a+1}} + \frac{a \times 2\beta - B}{2 br^{a+1}} \dots$$

oldest and *C* }
$$S, a, b, c, r, P, A$$
 and x representing

youngest. } the same quantities as in Theorem

3d, *B* the value of an Annuity cer-
tain for $\overline{b-a}$ years, and β the value
of a life whose complement is $\overline{b-a}$.

It would take up too much time to go through the separate investigations of those Theorems; though perhaps it may not be proper intirely to omit them. I shall, therefore, give the investigation of the 4th Theorem, which comprehends in it the chief reasoning that has been applied to the solution of the three foregoing ones.

INVESTIGATION OF THEOREM 4th.—By proceeding in the same manner with Mr. *Simpson* and retaining his symbols, the expectation during the time x is defined by the fluent $\frac{Sq^x}{2 abcrm} \times$

$x^2 - \frac{2x}{m} + \frac{2}{m^2}$, which being corrected (by first supposing x to vanish, and then to become equal to (a) ,) is changed into $\frac{aSq^a}{2 bcmr} - \frac{Sq^a}{bcm^2r} + \frac{S}{abcm^2r}$

$\times \overline{q^a - 1}$, which gives the whole value of the expectation for the first (a) years.—Let P be now substituted for its equal $-\frac{1}{m}$, C for $\overline{1 - q^a} \times -$

$\frac{1}{m}$, and x for q^x , then will the above expression be

at last reduced to $\frac{S}{2abc} \times \overline{C - xa} \times 2P^x - P^x a^2$.

The value of the expectation after the time (a) depends on the contingency of C's dying *after* the greatest limit of A's life, having previously survived B.—Now as B may have died *within* the limit of A's life, though *after* the decease of A, this probability must be taken into consideration, which (by reasoning as Mr. Simpson has done page 316 of his Select Exercises) will be expressed by $\frac{a}{2b}$. The probability of B's dying in x time after this period is $\frac{x}{b}$. These two fractions added

together give $\frac{a + 2x}{2b}$ for the *whole* probability of

B's dying in $a + x$ time according to the order in the Problem. The value, therefore, of the expectation on the death of C, during the moment x and under these circumstances, will be properly represented by $\frac{a + 2x}{2b} \times \frac{x}{c} \times Sq^x = \frac{aS}{2bc} \times q^x x +$

$\frac{S}{bc} \times q^x x x$ ---- The fluent of the first expression generated while x is increasing from a to b will be $\frac{aS}{2bc} \times B^*$. The 2d expression may be changed

* See Simpson's Select Exercises, page 324.

into

into $\frac{b-a}{bc} \cdot S \times \frac{q_x^{xx}}{b-a}$, whose fluent generated as

above is equal to $\frac{b-a}{bc} \cdot S \times \overline{B-\beta}^*$. As this

Reversion, expressed by both the preceding quantities, is not to take place till (a) years hence, and from the nature of the computation requires to be corrected by Dr. Price's rule †, it must be discounted for $a+1$ years, and consequently will be

reduced to $\frac{aS \times B}{2 bcr^{a+1}} + \frac{b-a}{bc} \cdot S \times \frac{\overline{B-\beta}}{r^{a+1}}$ -----

The third and last value of the expectation arises from the chance of C's not dying till after the extinction of both A and B. ----- This expectation

during the moment x will be $Sq^x \times \frac{x}{c}$, whose flu-

ent, generated while x is increasing from b to c , will be the value of the Reversion, and is equal to

$\frac{S}{c}$ drawn into M, the value of an Annuity certain

for $c-b$ years ‡. This being discounted for b years and corrected by Dr. Price's Rule, as in the case of the preceding expectation, will be expressed

by $\frac{S}{c} \times \frac{M}{r^{b+1}}$. Hence the whole value of all the

expectations is $\frac{S}{2 abcr} \times \overline{C-\alpha a} \times 2P^a - P\alpha a^2 +$

* See *Simpson's Select Exercises*, page 327.

† See *Treatise on Reversionary Payments*, Scholium to Question 10th.

‡ See *Simpson's Select Exercises*, page 324.

$$\frac{aS}{2bc} \times \frac{B}{r^{a+1}} + \frac{\overline{b-a}.S}{bc} \times \frac{\overline{B-\beta}}{r^{a+1}} + \frac{S}{c} \times \frac{M}{r^{b+1}}$$

which, by substituting $\frac{A}{r^{a+1}}$ for its equal $\frac{B}{r^{a+1}} +$

$$\frac{M}{r^{b+1}}, \&c. \text{ is at last reduced to } \frac{S}{2abc} \times \frac{B}{C - xa}$$

$$\times 2P^2 - Pxa^2 + \frac{S}{c} \times \frac{A - \beta}{r^{a+1}} + \frac{a.2\beta - B}{2br^{a+1}} \dots$$

Q. E. D.

The values by those Theorems in most cases agree nearly with the same values found by the Rules in page 175, &c. 1st. Let the given sum be 1000*l.* the ages as in the several examples of Problem 29th, and let C be the *oldest* life. Then by the first Theorem the value of the Reversion will be equal to 10.808*l.* which by the algebraical solution is determined to be 14.08*l.*—2dly, Let B be the *oldest* life. Then by proceeding according to Theorem 2d, the reversionary value will be found equal to 31.913*l.* which by the algebraical solution is 36.035*l.*—3dly, Let A be the *oldest* and C the *youngest* life. The value of the Reversion by Theorem 3d is = 52.411*l.* which by the algebraical solution is = 56.577*l.*—Lastly, Let A be the *oldest* and B the *youngest*. The reversionary value by Theorem 4th is = 65.56*l.* and the same value by the algebraical solution is = 67.881*l.*

It may be observed that the values of the Reversions by the fluxionary solutions are considerably *less* than those by the algebraical ones. The reason is,

is, that in computing from the former as well as the latter, the values of the Life-Annuities and Annuities certain have been taken on the supposition that they are payable *yearly*, whereas the fluxional solution supposes that they are payable *momently*.—Had the values of Annuities payable *momently* been taken, the results from both methods of solution would have been the same.—But if the values of Annuities payable *momently* are taken in computing from the fluxional solution, and of Annuities payable *yearly* in computing from the algebraical solution, the results will be always *less* from the former than from the latter; because the values of Annuities payable *momently* being greater than the values of Annuities payable *yearly*, the values of estates or sums of money to be received after their extinction, must be proportionably less.—All this is true, supposing the values of Annuities payable *momently* or *yearly* are given *accurately*. But if an approximation is used (as hath been done in the preceding examples in finding by Problem 6th the values of Annuities on the longest of three lives) another cause of difference will arise, which will be more or less as the approximation comes more or less near to exactness.—In order to illustrate all this, I shall beg leave to give the following examples, where the life and certain Annuities are supposed to be payable *momently*, and the values of Annuities on the longest of three lives are determined *exactly*.

EXAMPLE FIRST. Let the age of A be 20, that of B 30, and that of C 60 years. Let the

R

sum

sum be 100*l.* and, for greater perspicuity, let the decrements of life also be according to *De Moivre's* hypothesis.—This Case belongs to the first fluxional Theorem, where *a*, *b*, and *c* are respectively

equal to 66, 56, and 26; hence $\frac{S}{2abc} = \frac{100}{192192}$

----- A may be found = 16.167 --- *p* = .353; therefore, *P* being equal 25, $A - pc \times 2P^2$ is = 8736.25; and Ppc^2 is = 5965.7. Consequently

$$\frac{S}{2abc} \times A - pc \times 2P^2 - Ppc^2 = \frac{100}{192192} \times \frac{8736.25 - 5965.7}{1.04} = 1.381, \text{ which is the value re-}$$

quired.

Now let the same value be investigated by Rule 1st, Problem 29th. --- An Annuity on the longest of three equal lives aged 60 may be found equal to 13.16, which subtracted from 25 (the perpetuity) and the remainder being divided by 6, becomes = 1.9733. Therefore as ($56 \times 66 =$) 3696 is to ($26 \times 26 =$) 676 so is 1.9733 to .3609; which multiplied into 100, and the product divided by 26, we have 1.38 *l.* for the answer as above.

EXAMPLE SECOND. Let the age of A be 20, that of B 60, and that of C 30 years. The sum 100*l.*—This Case belongs to the 2d fluxional Theorem, where, by proceeding as in the 1st example, $\frac{S}{2abc} \times B - \pi b \times 2P^2 - P\pi b^2$ will be found equal to 1.38. N (the value of an Annuity certain

certain for 30 years) is ≈ 17.47 , and $\frac{N}{r^{t+1}} =$

$17.47 \times .34 = 5.9398$. Therefore $\frac{\delta S}{2ac} \times \frac{N}{r^{t+1}} =$

is $= \frac{2600}{7392} \times 5.9398 = 2.09$, which added to

1.38, found above, gives 3.47 *l.* for the value required.

Let this value be now sought by Rule 2d, Problem 29th.—By following the same steps as in the preceding example the $\frac{1}{6}$ th remainder will be

found ≈ 1.9733 . 12.5 (half the perpetuity) multiplied into .106 (the value of 1 *l.* at the end of 56 years) is ≈ 1.325 ----- As, therefore, 3696 is to 676 so is $(1.9733 - 1.325 =) .648$ to .1186, which must be *reserved*. ----- The value of an Annuity on the life of C for 30 years is ≈ 13.712 , which subtracted from 25 (the perpetuity) and divided by 2 quotes 5.644. Hence as 33 is to 13 so is 5.644 to 2.223, which multiplied into .353 (the value of 1 *l.* at the end of 26 years) is reduced to .7847, and this added to .1186 (*reserved* above) gives .9033, which being again multiplied into 100 and then divided by 26, we have 3.47 *l.* for the value required, as in the fluxional solution.

By proceeding in the same manner, the two other Theorems will be found to agree exactly with the corresponding Rules in Problem 29th. It will take up too much time to go through these operations; but the values resulting from them, as well as those investigated above, and also the

same values computed from different rates of mortality appear, when compared with each other, to be as follow.

EXACT Values of 100 l. supposing Annuities payable
MOMENTLY*.

By the fluxional Rule and <i>De Moivre's</i> Hypothesis.					By the algebraical Rule and <i>De</i> <i>Moivre's</i> Hypo- thesis.				
Cafe 1st,	-	-	-	1.38	-	-	-	1.38	
Cafe 2d,	-	-	-	3.47	-	-	-	3.47	
Cafe 3d,	-	-	-	5.41	-	-	-	5.41	
Cafe 4th,	-	-	-	6.68	-	-	-	6.68	

Values of 100 l. supposing Annuities payable YEARLY.

By the flux- ional Rule and the Hypothe- sis.	By the alge- braical Rule, the Hypothesis and the Ap- proximation in Problem 6th.	By the alge- braical Rule, the <i>Northamp-</i> <i>ton</i> Table, and the Approx- imations in Problems 2d, 3d, and 6th.	By the alge- braical Rule, the <i>Northamp-</i> <i>ton</i> Table, and taking the ex- act Values of the Lives.
Cafe 1st, - 1.08 -	1.40 -	1.381 -	1.384
Cafe 2d, - 3.19 -	3.60 -	3.531 -	3.524
Cafe 3d, - 5.24 -	5.65 -	5.513 -	5.429
Cafe 4th, - 6.55 -	6.78 -	6.753 -	6.680

* The Value of an Annuity on the *longest* of three lives has been found by the Rule in Problem 2, Corollary 2, *Simpson's Annuities*. The values of the *joint* lives (necessary for determining those on the *longest* of three lives) have also been found by a fluxional Rule given by the same author in his *Select Exercises*, page 325.—And the values of the *single* life, and certain Annuities have been found by Dr. Price's Theorems in the Philosophical Transactions, Vol. 66, No. 6.

Values

Values of 100 l. supposing Annuities payable momentarily in computing by the fluxional Rule, and that in computing by the algebraical Rule they are payable yearly and the Approximation in Problem 6th is used.

	By the fluxional Rule and De Moivre's Hypothesis.	By the algebraical Rule and De Moivre's Hypo- thesis.
Case 1st - - -	1.38 - - -	1.40
Case 2d, - - -	3.47 - - -	3.60
Case 3d, - - -	5.41 - - -	5.65
Case 4th, - - -	6.68 - - -	6.78

These values are set down in this order to give a clearer idea of what has been said in the preceding pages.—I shall only observe further on this subject, that the *algebraical* Rules are generally, if not always, preferable, as being the most simple; and that the values of the Life and certain Annuities may be taken as payable *yearly* without any material error, even though the Reversions are in *landed estates*, which are the only cases where the *fluxional* solutions ought to be applied.

Note (I). PROBLEMS XXX, XXXI, &c.

Pag. 185, &c.

PROBLEM XXX. Let L , n , and l signify the numbers of the living in the Table at the ages of A , B , and C ----- D' , D'' , D''' , &c. the decrements of life in the Table at the end of 1, 2, 3,

R 3

&c.

&c. years from the age of A ---- $\Delta', \Delta'', \Delta''', \&c.$ and $d', d'', d''', \&c.$ the decrements from the ages of B and C ---- $e, f, g, \&c.$ and $o, p, q, \&c.$ the numbers of the living in the Table after 1, 2, 3, &c. years from the ages of C and B respectively. ---- Let r also denote 1 $l.$ increased by its interest for a year, S the given sum, and t the given number of years. --- Then by reasoning as Mr. Simpson has done in pag. 321, &c. of his *Select Exercises*, the value of B's expectation will be expressed by

$$\begin{aligned} & \frac{S \times o}{n} \times \frac{D' \times d'}{2Llr} + \frac{S \times p}{n} \times \frac{d' \times D''}{Llr^2} + \\ & \frac{d'' \times D''}{2Llr^2} + \frac{S \times q}{n} \times \frac{d' + d'' \cdot D''}{Llr^3} + \frac{d''' \times D'''}{2Llr^3} \\ & \&c. (t) = \frac{S}{L} \times \frac{o \cdot D'}{nr} + \frac{p \cdot D''}{nr^2} + \frac{q \cdot D'''}{nr^3} \&c. \\ & \text{---} (t) = \frac{S}{L} \times \frac{l - d' \times o D'}{nlr} + \frac{l - d' + d'' \times p D''}{nlr^2} \\ & + \frac{l - d' + d'' + d''' \times q D'''}{nlr^3}, \&c. \text{---} (t) = \frac{S}{2Lln} \\ & \times \frac{o \cdot d' \cdot D'}{r} + \frac{p \cdot d'' \cdot D''}{r^2} + \frac{q \cdot d''' \cdot D'''}{r^3}, \&c. (t) \end{aligned}$$

which is equal nearly to $\frac{S}{a} \times \overline{N-M} - \frac{S}{2ac}$
 $\times N^*$, ---- M representing the value of the joint
 lives

* The last term in this Theorem, or $\frac{S}{2ac} \times N$, is of
 no consequence when the condition of Survivorship is extended

lives of B and C for t years, N the value of the life of B for the same term, and a and c the complements of the lives of A and C for t years *.

This Theorem will give answers sufficiently correct if the term is not less than five or six years, and the values of the single and joint lives can be obtained true to three places of decimals. But if, for want of proper tables, such values cannot be procured, and the term is short, it will be necessary and it will also be easiest to find the answer from the series itself rather than the Theorem. For instance, were the term *three years* in the example to this Problem instead of 16 years, the first series in the preceding investigation taken as far as I have carried it will express the *exact* value †; and it

tended through the whole duration of the lives, or any great number of years; but when it is limited to a small number of years, this term becomes considerable and should by no means be omitted.

* If a and c are made to signify the complements of the lives of A and C for their *whole duration*, the answers may in some cases be a *quarter* or even a *third* wrong.

† The terms of the series in the Example to the Problem, supposing the number of years to be only *three*, and the decrements of life as they are in Table 3d, will

$$\begin{aligned} \text{be } & \frac{1000 \times 428}{1.04 \times 435} \times \frac{8 \times 9}{2 \times 284 \times 365} + \frac{1000 \times 421}{1.04^2 \times 435} \times \\ & \frac{8 \times 8}{284 \times 365} + \frac{8 \times 8}{2 \times 284 \times 365} + \frac{1000 \times 414}{1.04^3 \times 435} \times \\ & \frac{16 \times 8}{284 \times 365} + \frac{8 \times 8}{2 \times 284 \times 365} = .3285 + .8285 + \end{aligned}$$

1.306 = 2.453 l. - - - The same value by the approxi-

it will be easier to obtain an answer by finding the number it expresses, than by making use of the Rule.

These observations are to be applied to all the solutions of Problems for determining the values of Reversions depending on Survivorships within a given time,

PROBLEM THIRTY-FIRST.—Page 188.—Let the same symbols be retained, and P denote the perpetuity. Let A, B, and C also signify the probabilities respectively that A, B and C shall live t years. — The values of the 1st, 2d, 3d, &c. years rents in this Problem may be considered as depending on one or other of the four following events. — First, that A is dead, and B and C are both alive at the beginning of the 2d, 3d, 4th, &c. years. — Secondly, that A and B are both dead (A having died first) and C alive at the beginning of those years. Thirdly, that A and C are both dead (A having died first) and B alive at the beginning of those years. Lastly, that A, B, and C are all dead at the beginning of those years, A having died first.—By reasoning in the same manner as hath been done in investigating the 23d and the following Problems, the values of the several rents for t years and ever afterward on the *first* contingency
 mations in Problems 2d and 3d, and computing agreeable to the general Rule, may be found equal to 2.597/. Those answers agree with each other very nearly in the present instance, owing to the decrements of life being *equal* in the Table. When this is not the case, the difference in so small a term will be much more considerable,

will

will be expressed by the series $\frac{e}{lr} \times \frac{o}{n} \times \frac{D}{L}$
 $+ \frac{f}{lr^2} \times \frac{p}{n} \times \frac{D' + D''}{L} + \frac{g}{lr^3} \times \frac{q}{n} \times$
 $\frac{D' + D'' + D'''}{L} \&c. (1) + \frac{B \times C \times 1 - A}{r^2} \times P$

----- The values of the several rents for t years
 and ever afterward on the *second* contingency will
 be expressed by the series $\frac{e}{lr} \times \frac{D' \times \Delta'}{2Ln} +$

$\frac{f}{lr^2} \times \frac{D' + D'' \times \Delta' + \Delta''}{2Ln} + \frac{g}{lr^3} \times$
 $\frac{D' + D'' + D''' \times \Delta' + \Delta'' + \Delta'''}{2Ln} \&c. \dots (1)$

$+ \frac{C \times 1 - A \times 1 - B \times P}{2r^2} \dots \dots \dots$ On the

third contingency by $\frac{o}{nr} \times \frac{D' \times d'}{2Ll} + \frac{p}{nr^2}$

$\times \frac{D' + D'' \times d' + d''}{2Ll} + \frac{q}{nr^3} \times$

$\frac{D' + D'' + D''' \times d' + d'' + d'''}{2Ll} \&c. (1) +$

$\frac{B \times 1 - C \times 1 - A}{2r^2} \times P. \dots$ On the *fourth* contingency

by $\frac{D' \times d' \times \Delta'}{3nLlr} + \frac{D' + D'' \times d' + d'' \times \Delta' + \Delta''}{3nLlr^2} +$

$\frac{D' + D'' + D''' \times d' + d'' + d''' \times \Delta' + \Delta'' + \Delta'''}{3nLlr^3}$

$\&c. (1) + \frac{1 - A \times 1 - B \times 1 - C}{3r^2} \times P.$

The

The sum of all these serieses expresses the *exact* value according to any Table of observations, and gives the only just solution when the term does not exceed five or six years. When the term is longer, the following approximation will give answers sufficiently accurate.

Let a , b , and c represent, as before, the complements of the lives of A, B, and C for the given term agreeable to any Table of observations. Then the value of the several rents for t years and ever afterward will be nearly according to that Table;

--- On the *first* contingency $\frac{\overline{b-1} \cdot \overline{c-1} \cdot 1}{abc} +$
 $\frac{\overline{b-2} \cdot \overline{c-2} \cdot 2}{abc^2} + \frac{\overline{b-3} \cdot \overline{c-3} \cdot 3}{abc^3} + \&c. (t)$
 $+ \frac{\overline{b-t} \cdot \overline{c-t} \cdot t}{abc^t} \times P. \dots$ On the *second* con-
 tingency $\frac{\overline{c-1} \cdot 1^2}{2abc} + \frac{\overline{c-2} \cdot 2^2}{2abc^2} + \frac{\overline{c-3} \cdot 3^2}{2abc^3} +$
 $\&c. (t) + \frac{\overline{c-t} \cdot t^2}{2abc^t} \times P. \dots$ On the *third*
 contingency $\frac{\overline{b-1} \cdot 1^2}{2abc} + \frac{\overline{b-2} \cdot 2^2}{2abc^2} + \frac{\overline{b-3} \cdot 3^2}{2abc^3}$
 $+ \&c. (t) + \frac{\overline{b-t} \cdot t^2}{2abc^t} \times P. \dots$ On the *fourth*
 contingency $\frac{1^3}{3abc} + \frac{2^3}{3abc^2} + \frac{3^3}{3abc^3} \&c.$
 $\dots (t) + \frac{t^3}{3abc^t} \times P. \dots$ If all these
 serieses be resolved and added together, they will
 be

be transformed into $\frac{1}{ar} + \frac{2}{ar^2} + \frac{3}{ar^3} \dots (t)$

$$+ \frac{t}{ar^t} \times P - \frac{b+c}{2abc} \times \frac{1^3}{r} + \frac{2^3}{r^2} + \frac{3^3}{r^3} \dots (t)$$

$$+ \frac{tt}{r^t} \times P + \frac{1}{3abc} \times \frac{1^3}{r} + \frac{2^3}{r^2} + \frac{3^3}{r^3} \&c.$$

(t) $+ \frac{t^3}{r^t} \times P \dots$ The first series is the same

with what Dr. Price has given in Note (G) of the Appendix to his Treatise on Reversionary Payments, for determining the value of an Annuity after the decease of A, should it happen in a given term, and which has been also given in words in the solution of the 9th Problem in the preceding chapter of this work. ----- Let this series, therefore, be denoted by R, the value of an Annuity on the longest of two equal lives (whose complement is *t*) by D, and the value of an Annuity on the longest of three such equal lives by T; then the whole value of the Reversion will be

expressed by $R - \frac{b+c \times tt}{2abc} \times P - D + \frac{t^3}{3abc} \times$

$$P - T = R - \frac{b+c}{2} \times P - D - \frac{t}{3} \times P - T$$

$\times \frac{tt}{abc}$, which is the general rule in page 188.

Having given these specimens, it can hardly be necessary to proceed through the investigations of the following Problems: I shall, therefore, give the serieses expressing the *exact* value according to any Table of observations, and also the general theorem

theorem or *approximation*, without entering into any particular detail.

PROBLEM THIRTY-SECOND, page 192.--- The *exact* answer to this Problem is expressed by $\frac{e}{lr}$

$$\begin{aligned} & \times \frac{D' \times \Delta'}{2Ln} + \frac{f}{lr^2} \times \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''}}{2Ln} + \\ & \frac{g}{lr^3} \times \frac{\overline{D' + D'' + D'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{2Ln} \&c. (1) \\ & + \frac{C \times \overline{1-A} \times \overline{1-B}}{2r^2} \times P \dots\dots + \frac{o}{nr} \\ & \times \frac{D' \times d'}{2Ll} + \frac{p}{nr^2} \times \frac{\overline{D' + D''} \times \overline{d' + d''}}{2Ll} + \\ & \frac{q}{nr^3} \times \frac{\overline{D' + D'' + D'''} \times \overline{d' + d'' + d'''}}{2Ll} \&c. (2) \\ & + \frac{B \times \overline{1-A} \times \overline{1-C}}{2r^2} \times P \dots\dots + \\ & \frac{D' \times \Delta' \times d'}{3Lnr} + \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{3Lnr^2} \\ & + \&c. (3) + \frac{\overline{1-A} \times \overline{1-B} \times \overline{1-C}}{3Lnr^2} \times P. \end{aligned}$$

----- The *approximation* as given in page 192

$$\begin{aligned} & \text{may be found equal to } \frac{tt}{abc} \times \frac{\overline{b+c}}{2} \times \overline{P-D} \\ & = \frac{2t. \overline{P-T}}{3}. \end{aligned}$$

PROBLEM

PROBLEM THIRTY-THIRD, page 193. -----

The *exact* answer by any Table is expressed by

$$\frac{D' \times \Delta' \times d'}{3 n L r} + \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{3 n L r^2} +$$

$$\frac{\overline{D' + D'' + D'''} \times \overline{d' + d'' + d'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{3 n L r^3}$$

$$\&c. (t) + \frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{3 r^t} \times P.$$

$$\text{The general theorem or approximation is} = \frac{t^3}{3 abc} \times \overline{P - T}.$$

PROBLEM THIRTY-FOURTH, page 195. -----

The *exact* value is expressed by $\frac{e}{lr} \times \frac{D' \times \Delta'}{Ln}$

$$+ \frac{f}{ln^2} \times \frac{D' + D'' \times \overline{\Delta' + \Delta''}}{Ln} + \frac{g}{lr^3} \times$$

$$\frac{D' + D'' + D''' \times \overline{\Delta' + \Delta'' + \Delta'''}}{Ln} \&c. (t) +$$

$$C \times \frac{\overline{1 - A} \times \overline{1 - B}}{r^t} \times P + \frac{D' \times \Delta' \times d'}{3 Ln l r}$$

$$+ \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{3 Ln l r^2} \&c. (t) +$$

$$\frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{3 r^t} \times P \text{ ----- The}$$

general theorem or *approximation* may be found

$$\text{equal to } \frac{tt}{ab} \times \overline{P - D} - \frac{2 t^3}{3 abc} \times \overline{P - T}.$$

PROBLEM

PROBLEM THIRTY-NINTH, page 201. --- The *exact* answer to this Problem is expressed by the single series $\frac{D' \times \Delta' \times d'}{6Lnlr} + \frac{D' + D'' \times \Delta' + \Delta'' \times d' + d''}{6Lnlr^2}$ &c. (1) + $\frac{1-A \times 1-B \times 1-C}{6r^2} \times P \dots$

and the *approximation* to the real value by $\frac{t^3}{6abc} \times P - T$ which is just *half* the value of the Reversion by Problem 33d.

PROBLEM FORTIETH, page 202. --- The series expressing the answer to this Problem is $\frac{1}{2Ll} \times \frac{o \times D' \times d'}{nr} + \frac{p \times D' + D'' \times d' + d''}{nr^2}$ &c. (1) + $\frac{1-A \times 1-C \times \frac{V}{2}}{1}$; V denoting the value of the life of B after *t* years.

The general theorem or *approximation* may be found = $\frac{tt}{2ac} \times M - D$; M representing the value of an Annuity on the longest of three lives, one of whom being of the age of B, and the two others of a life whose complement is *t*, and whose decrements are equal.

COROLLARY TO PROBLEM FORTIETH. --- The *exact* answer is evidently the first part of the series given above, or $\frac{1}{2Ll} \times \frac{o \times D' \times d'}{nr} +$

$\frac{p \times D' + D'' \times d' + d''}{nr^2}$ ----- (t), and the *approximation* to this value may be found equal to $\frac{tt}{ac} \times \frac{\tau - D}{2} - \frac{t \times \tau - T}{2b}$; τ denoting the value of an Annuity certain for t years.

PROBLEM FORTY-FIRST, page 205. ----- The value of this Annuity being evidently the value in Problem 40th, subtracted from the Reversion on the life of B *after* A, provided B should survive A in the given time; the only thing necessary for the solution of this Problem is to find the value of such a Reversion. ----- The series expressing the value of this Reversion is $\frac{o}{nr} \times \frac{D'}{L} + \frac{p}{nr^2} \times \frac{D' + D''}{L} + \frac{q}{nr^3} \times \frac{D' + D'' + D'''}{L}$ &c. (t) $+ 1 - A \times V$, which, therefore, being lessened by the value found from the *series* in Problem 40th, will give the answer in *exact* agreement to any Table of observations; V denoting the value of the life of B after t years as in that Problem.

The *approximation* to the real value of the Reversion just mentioned (and which is to be lessened by the value found from the *theorem* in Problem 40th) is equal to $\frac{t}{a} \times \overline{B - \beta}$, where B represents the value of the life of B, and β the value of an Annuity on the joint continuance of B and another

ther life whose complement is t , and whose decrements are equal.

PROBLEM FORTY-SECOND, page 208. --- The serieses expressing the *exact* value in this Problem

$$\begin{aligned} \text{are } & \frac{e}{lr} \times \frac{D' \times \Delta'}{Ln} + \frac{f}{ln^2} \times \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''}}{Ln} \\ & + \frac{g}{lr^3} \times \frac{\overline{D' + D'' + D'''} \times \overline{\Delta' + \Delta'' + \Delta'''}}{Ln} \&c. (t) \\ & + \frac{C \times \overline{1 - A} \times \overline{1 - B}}{r^2} \times P \text{ ----- and} \\ & \frac{D' \times \Delta' \times d'}{2 Ln r} + \frac{\overline{D' + D''} \times \overline{\Delta' + \Delta''} \times \overline{d' + d''}}{2 Ln r^2} \\ & \&c. (t) + \frac{\overline{1 - A} \times \overline{1 - B} \times \overline{1 - C}}{2 r^2} \times P. \end{aligned}$$

The general theorem, or *approximation* to the exact value, is $\frac{tt}{ab} \times \overline{P - D} - \frac{t^3}{2 abc} \times \overline{P - T}$.

Note

Note (K).

An Account of the Method by which the Values in Annual Payments have been deduced.

PROBLEM XVIII. page 144. - - - - This claim will be undetermined as long as the three joint lives A, B, and C continue, and also during all that may happen to remain of the joint lives of A and B after C. - - - - Now the last of these values is found by subtracting the value of the three joint lives A, B, and C from the joint lives in expectation; that is, from the joint lives of A and B - - - - Add to this the former value, and the result will give the general rule. - - - - Let $A B C$, $A C$, $B C$, $A B$, constantly represent the values of the several joint lives combined as those letters are; and let $\overset{1}{A} \overset{1}{C}$, $\overset{1}{B} \overset{1}{C}$, $\overset{1}{A} \overset{1}{B} \overset{1}{C}$, &c. represent the values of the *longest* of two or three lives. Then in the present case we shall have $AB - ABC + ABC = AB$ for the divisor. - - - - Should the money *become payable*, it will become so at the same time that the *claim is determined*; and, therefore, the annual payments in both cases are the same.

The like may be observed in all the following solutions, where no mention is made of any difference in those values.

PROBLEM XX. page 149. - - - - This claim cannot be determined till after the extinction of the joint lives of A and C, and likewise the joint lives of A and B. In other words, it will be undetermined

S

mined

mined while the joint lives of A and C continue; and also during the whole time that the joint lives of A and B may happen to continue after C. The last of these values is equal to $\overline{AB} - \overline{ABC}$, which added to the first value, or AC, we have $AC + \overline{AB} - \overline{ABC}$ for the required divisor.

PROBLEM XXI. page 151. ---- The time when the claim is *determined* is found just in the same manner as in the preceding Problem, and consequently the divisor for ascertaining the annual payments in this case will be also $= AC + \overline{AB} - \overline{ABC}$.

Should the claim, however, be determined in favour of the purchaser, it is evident that the money will not become payable till after a further time, and, therefore, under these circumstances the annual payments ought to be *less*; for they must be continued not only during the joint lives of A and C and the remainder of the joint lives of A and B after C, but likewise during all that may happen to remain of the single life A after the longest of the two lives C and B. Now the value of an Annuity on the last of these Reversions is $= \overline{A} \overline{B} \overline{C} - \overline{B} \overline{C}$, which being resolved * and then added to $AC + \overline{AB} - \overline{ABC}$, given above, will at

* By Scholium to Problem 6th, $\overline{A} \overline{B} \overline{C} = A + B + C + \overline{ABC} - \overline{AB} - \overline{AC} - \overline{BC}$ ---- By the Rule in page 55, $\overline{B} \overline{C} = B + C - \overline{BC}$; hence $\overline{A} \overline{B} \overline{C} - \overline{B} \overline{C} = A + \overline{ABC} - \overline{AC} - \overline{AB}$, which being added to $AC + \overline{AB} - \overline{ABC}$, we have A for the required divisor.

last

last be found equal to an Annuity on the single life of A.

PROBLEM XXII. page 153: ----- Let A be always supposed to be the youngest of the two lives A and B: ----- In this case the claim will be undetermined during the joint lives of B and C, and also during the continuance of the joint lives of A and C after B. Hence the required divisor is equal to $BC + AC - ABC$:

PROBLEM XXIII. page 154. ---- It is evident that the claim is *determined* here at the extinction of the three joint lives. But the annual payments till the claim *becomes due*; that is, till the claim is determined and afterwards till it becomes payable, provided it should be determined in favour of the purchaser, will be much *less* than the preceding ones, since they must be continued not only during the joint lives of A, B and C, but also during the remainder of the longest of the two lives A and B after C. ---- The value of an Annuity on such a Reversion is equal $\overset{1}{A}\overset{1}{B}\overset{1}{C} - C$; therefore the divisor in this case will be $= ABC + \overset{1}{A}\overset{1}{B}\overset{1}{C} - C$.

PROBLEM XXIV. page 161. ---- The claim in this Problem cannot be determined till after the extinction of the joint lives of A and C, and all that may happen to remain of the joint lives of A and B after C. Consequently the required divisor will be $= AC + AB - ABC$.

PROBLEM XXV. page 163. - - - The time in which the claim, in this case, is *determined*, may be found by reasoning in the same manner as in Problems 20th and 21st. Therefore the general divisor will also be the same; that is, equal to $AC + AB - ABC$.

The time in which the claim *becomes payable* may likewise be found as in Problem 21st. But in that, as well as in the present Problem, this time may be ascertained in a much shorter way; for since it is evident that the claim cannot be determined *against* a purchaser by any other death than that of A, it follows, by making the annual payments during A's life, that a purchaser cannot become bound to make any payments *after* the claim has been determined against him.

PROBLEM XXVI. page 164. - - - The *determination* of the claim in this Problem depends on the same contingencies with those in Problem 20th; for it is plain that whenever it is determined that A *will* or *will not* be the *second* that dies, it will at the same time be determined that he *will not* or *will* be either the *first* or *last* that dies.

The same reasoning may be applied, towards finding the value in *annual* payments till the money *becomes due*, as in Problem 25th.

It may not be improper in this place to add the following investigation; which will further confirm the Rule given in Problems 20th, 21st, 25th, and 26th, for finding the value in annual payments
till

till the claim is *determined*. ----- Let the present, or 26th, Problem be taken. ----- The title to the 1st, 2d, 3d, &c. payments of an Annuity to be continued till the claim, specified in this Problem, is *determined*, depends on three different contingencies. — 1st, On A, B, and C being all living at the end of the 1st, 2d, 3d, &c. years. Secondly, On A and B being living and C dead at the end of the 1st, 2d, 3d, &c. years. Thirdly, On A and C being living and B dead at the end of the 1st, 2d, 3d, &c. years. The value, therefore, of the Annuity, as far as it depends on the *first* of these events, is the value of the three joint lives. The value of the Annuity, as far as it depends on the *second* event, is the value of the two joint lives A and B after C; that is, the value of the two joint lives A and B lessened by the value of the three joint lives. The third value is the value of the two joint lives A and C after B, or the value of the joint lives A and C lessened by the three joint lives. The sum of all these values, or $AC + AB - ABC$, is the rule.

PROBLEM XXVIII. page 173. ----- The claim in this Problem will be *determined* at the extinction of the joint lives of A and C; but the money will not *become payable* till after the *longest* of the two lives A and B; nor can the purchaser even then be entitled to any claim, unless A shall have died in the life-time of C. In order, therefore, to obtain a proper divisor in this case, it will be necessary to add to the value of the joint lives

A and C the value of the life of B after A, provided C shall have survived A. ---- The last of these values is given in Problem 27th, which has been investigated partly with a view to the solution of this Problem.

PROBLEM XXIX. page 175. ---- The method of finding the time in which the claim is here *determined*, is nearly the same with that which has been used in Problem 18th. For an Annuity to be paid till this event is known must be paid during the three joint lives, and also during the joint lives of B and C after A ; that is, the value of such an Annuity is equal to $ABC + BC - ABC = BC$. But as the purchaser cannot come into *possession* of the estate (if the claim is determined in his favour) till after the extinction of the last surviving life C ; it follows that the value of an Annuity to be continued to this period must be equal to the former value increased by the value of the life of C after B, provided B survives A, which is found by the Corollary to Problem 27th*.

* The rules, for finding the values of two lives after one, of one life after two, and of two *joint* lives after one, which have been used in the preceding solutions, are the same with those in the 16th, 18th, and 19th Problems in Mr. *Simpson's* Select Exercises.

TABLE

T A B L E F I R S T.

The present Value of 1 l. to be received at the End of any Number of Years, not exceeding 92; discounting at the Rate of Four per Cent. Compound Interest.

Years.	Four per Cent.	Years.	Four per Cent.
1	.961538	28	.333477
2	.924556	29	.320651
3	.888996	30	.308319
4	.854804	—	—
5	.821927	31	.296460
6	.790315	32	.285058
7	.759918	33	.274094
8	.730690	34	.263552
9	.702587	35	.253415
10	.675564	36	.243669
—	—	37	.234297
11	.649581	38	.225285
12	.624597	39	.216621
13	.600574	40	.208289
14	.577475	—	—
15	.555265	41	.200278
16	.533908	42	.192575
17	.513373	43	.185168
18	.493628	44	.178046
19	.474642	45	.171198
20	.456387	46	.164614
—	—	47	.158283
21	.438834	48	.152195
22	.421955	49	.146341
23	.405726	50	.140713
24	.390121	—	—
25	.375117	51	.135301
26	.360689	52	.130097
27	.346817	53	.125093

APPENDIX.

Years.	Four per Cent.	Years.	Four per Cent.
54	.120282	74	.054895
55	.115656	75	.052784
56	.111207	76	.050754
57	.106930	77	.048801
58	.102817	78	.046924
59	.098963	79	.045120
60	.095060	80	.043384
—	—	—	—
61	.091404	81	.041716
62	.087889	82	.040111
63	.084508	83	.038569
64	.081258	84	.037085
65	.078133	85	.035659
66	.075128	86	.034287
67	.072238	87	.032968
68	.069460	88	.031700
69	.066788	89	.030481
70	.064219	90	.029309
—	—	—	—
71	.061749	91	.028182
72	.059374	92	.027098
73	.057091		

TABLE

TABLE SECOND.

The present Value of an Annuity of One Pound, for any Number of Years not exceeding 92; at the Rate of Four per Cent.

Years.	Years Purchase.	Years.	Years Purchase.
1	.9615	28	16.6630
2	1.8860	29	16.9837
3	2.7750	30	17.2920
4	3.6298	—	—
5	4.4518	31	17.5884
6	5.2421	32	17.8735
7	6.0020	33	18.1476
8	6.7327	34	18.4111
9	7.4353	35	18.6646
10	8.1108	36	18.9082
—	—	37	19.1425
11	8.7604	38	19.3678
12	9.3850	39	19.5844
13	9.9856	40	19.7927
14	10.5631	—	—
15	11.1183	41	19.9930
16	11.6522	42	20.1856
17	12.1656	43	20.3707
18	12.6592	44	20.5488
19	13.1339	45	20.7200
20	13.5903	46	20.8846
—	—	47	21.0429
21	14.0291	48	21.1951
22	14.4511	49	21.3414
23	14.8568	50	21.4821
24	15.2469	—	—
25	15.6220	51	21.6174
26	15.9827	52	21.7475
27	16.3295	53	21.8726

Years.	Years Purchase.	Years.	Years Purchase.
54	21.9929	74	23.6276
55	22.1086	75	23.6804
56	22.2198	76	23.7311
57	22.3267	77	23.7799
58	22.4295	78	23.8268
59	22.5284	79	23.8720
60	22.6234	80	23.9153
61	22.7148	81	23.9571
62	22.8027	82	23.9972
63	22.8872	83	24.0357
64	22.9685	84	24.0728
65	23.0466	85	24.1085
66	23.1218	86	24.1428
67	23.1940	87	24.1757
68	23.2635	88	24.2074
69	23.3302	89	24.2379
70	23.3945	90	24.2672
71	23.4562	91	24.2954
72	23.5156	92	24.3225
73	23.5727		

TABLE

TABLE THIRD.

*Shewing the Probabilities of the Duration of Life,
as deduced by Dr. PRICE from Observations on
the Bills of Mortality at NORTHAMPTON.*

Ages.	Persons living.	Decrem. of Life.	Ages.	Persons living.	Decrem. of Life.
1	849	127	30	435	7
2	722	50	31	428	7
3	672	26	32	421	7
4	646	21	33	414	7
5	625	16	34	407	7
6	609	13	35	400	7
7	596	10	36	393	7
8	586	9	37	386	7
9	577	7	38	379	7
10	570	6	39	372	7
11	564	6	40	365	8
12	558	5	41	357	8
13	553	5	42	349	8
14	548	5	43	341	8
15	543	5	44	333	8
16	538	5	45	325	8
17	533	5	46	317	8
18	528	6	47	309	8
19	522	7	48	301	8
20	515	8	49	293	9
21	507	8	50	284	9
22	499	8	51	275	8
23	491	8	52	267	8
24	483	8	53	259	8
25	475	8	54	251	8
26	467	8	55	243	8
27	459	8	56	235	8
28	451	8	57	227	8
29	443	8	58	219	8

APPENDIX.

Ages.	Persons living.	Decrem. of Life.	Ages.	Persons living.	Decrem. of Life.
59	211	8	76	75	8
60	203	8	77	67	7
61	195	8	78	60	7
62	187	8	79	53	7
63	179	8	80	46	7
64	171	8	81	39	7
65	163	8	82	32	6
66	155	8	83	26	5
67	147	8	84	21	4
68	139	8	85	17	4
69	131	8	86	13	3
70	123	8	87	10	2
71	115	8	88	8	2
72	107	8	89	6	2
73	99	8	90	4	2
74	91	8	91	2	1
75	83	8	92	1	1

TABLE

TABLE FOURTH.

*Shewing the Expectations of Life in NORTHAMPTON,
according to the preceding Table.*

Age.	Expectation.	Age.	Expectation.	Age.	Expectation.	Age.	Expectation.
1	33.29	24	31.26	47	19.38	70	8.42
2	38.06	25	30.78	48	18.87	71	7.97
3	39.86	26	30.30	49	18.38	72	7.53
4	40.44	27	29.82	50	17.94	73	7.10
5	40.78	28	29.34	51	17.51	74	6.68
6	40.84	29	28.86	52	17.02	75	6.28
7	40.72	30	28.38	53	16.53	76	5.90
8	40.41	31	27.83	54	16.05	77	5.54
9	40.03	32	27.29	55	15.56	78	5.13
10	39.52	33	26.74	56	15.07	79	4.74
11	38.93	34	26.19	57	14.58	80	4.39
12	38.34	35	25.65	58	14.10	81	4.09
13	37.69	36	25.09	59	13.61	82	3.87
14	37.02	37	24.54	60	13.13	83	3.65
15	36.36	38	23.98	61	12.65	84	3.40
16	35.69	39	23.42	62	12.17	85	3.08
17	35.02	40	22.86	63	11.69	86	2.88
18	34.35	41	22.37	64	11.24	87	2.60
19	33.74	42	21.87	65	10.74	88	2.12
20	33.19	43	21.37	66	10.27	89	1.66
21	32.71	44	20.87	67	9.80	90	1.25
22	32.22	45	20.37	68	9.34	91	1.00
23	31.74	46	19.87	69	8.88	92	.50

TABLE

TABLE FIFTH.

Shewing the present Value of an Annuity of 1 l. on a single Life at Four per Cent., according to Mr. De Moivre's Hypothēsis; and nearly according to the Probabilities of Life at NORTHAMPTON: See Table, Page 64.

Age.	Years Purchase.	Age.	Years Purchase.	Age.	Years Purchase.	Age.	Years Purchase.
8	16.791	28	14.946	48	11.748	68	6.714
9	16.882	29	14.816	49	11.548	69	6.394
10	16.882	30	14.684	50	11.344	70	6.065
—	—	—	—	—	—	—	—
11	16.791	31	14.549	51	11.135	71	5.728
12	16.698	32	14.411	52	10.921	72	5.383
13	16.604	33	14.270	53	10.702	73	5.029
14	16.508	34	14.126	54	10.478	74	4.666
15	16.410	35	13.979	55	10.248	75	4.293
16	16.311	36	13.829	56	10.014	76	3.912
17	16.209	37	13.676	57	9.773	77	3.520
18	16.105	38	13.519	58	9.527	78	3.111
19	15.999	39	13.359	59	9.275	79	2.707
20	15.891	40	13.196	60	9.017	80	2.284
—	—	—	—	—	—	—	—
21	15.781	41	13.028	61	8.753	81	1.850
22	15.669	42	12.858	62	8.482	82	1.406
23	15.554	43	12.683	63	8.205	83	0.950
24	15.437	44	12.504	64	7.921	84	0.481
25	15.318	45	12.322	65	7.631	85	0.000
26	15.197	46	12.135	66	7.333		
27	15.073	47	11.944	67	7.027		

TABLE

TABLE SIXTH*.

Shewing the present Value of an Annuity of 1 l. on the joint Continuance of two Lives at Four per Cent., according to Mr. De Moivre's Hypothesis; and nearly according to the Probabilities of Life at NORTH-AMPTON. See Table, Page 74.

Ages.		Years Purchase.	Ages.		Years Purchase.
10	10	13.342	15	70	5.623
	15	13.093	—	—	—
	20	12.808	—	20	12.341
	25	12.480	—	25	12.051
	30	12.102	—	30	11.711
	35	11.665	—	35	11.314
	40	11.156	—	40	10.847
	45	10.564	20	45	10.297
	50	9.871	—	50	9.648
	55	9.059	—	55	8.879
15	60	8.105	—	60	7.967
	65	6.980	—	65	6.882
	70	5.652	—	70	5.590
	—	—	—	—	—
	15	12.860	—	25	11.786
	20	12.593	—	30	11.468
	25	12.281	—	35	11.095
	30	11.921	—	40	10.655
	35	11.501	—	45	10.131
	40	11.013	25	50	9.509
20	45	10.440	—	55	8.766
	50	9.767	—	60	7.880
	55	8.975	—	65	6.826
	60	8.041	—	70	5.551
	65	6.934	—	—	—

* This is taken from Dr. Price's Table of the values of joint lives published in his *Treatise on Reversionary Payments*. See page 328, 3d edit.

Ages.	Years Purchase.	Ages.	Years Purchase.
30	30 11.182	45	45 9.063
	35 10.838		50 8.619
	40 10.428		55 8.044
	45 9.936		60 7.332
	50 9.345		65 6.425
	55 8.634		70 5.300
	60 7.779		— —
	65 6.748		50 8.235
35	70 5.505	50	55 7.738
	— —		60 7.091
	35 10.530		65 6.258
	40 10.157		70 5.193
	45 9.702		— —
	50 9.149		55 7.332
	55 8.476		60 6.781
	60 7.658		65 6.036
40	65 6.662	55	70 5.053
	70 5.450		— —
	— —		60 6.351
	40 9.826		65 5.730
	45 9.418		70 4.858
	50 8.911		— —
	55 8.283		65 5.277
	60 7.510		70 4.571
45	65 6.556	65	— —
	70 5.383		70 4.104

TABLE

TABLE SEVENTH*.

Shewing the present Value of an Annuity of 1 l. on the joint Continuance of Three equal Lives at Four per Cent., according to the Probabilities of Life in NORTHAMPTON. See Table 3d.

Common Age.	Years Purchase.	Common Age.	Years Purchase.
60	4.7826	76	1.9089
61	4.6115	77	1.7846
62	4.4382	78	1.5843
63	4.2626	79	1.3906
64	4.0849	80	1.2121
65	3.9050	81	1.0685
66	3.7230	82	1.0117
67	3.5390	83	0.9617
68	3.3533	84	0.8981
69	3.1662	85	0.7606
70	2.9780	86	0.7690
71	2.7895	87	0.7568
72	2.6015	88	0.5368
73	2.4160	89	0.3233
74	2.2352	90	0.1346
75	2.0636	91	0.1202

* This table has been computed agreeable to the method described in chap. II. sect. II. ; and as it was only designed for the purpose of comparing the values in some of the foregoing examples, (where the hypothesis and the approximations in Prob. 5th and 6th are used) with the *exact* values, it was thought unnecessary to extend it below the age of 60 ; especially as the hypothesis and approximations correspond so nearly with the truth under that age. See Schol. to Prob. 6th, page 102.

APPENDIX

TABLE SEVENTH

Showing the results of the experiments conducted at the Central Laboratory of the Ministry of Agriculture, in 1901, for the purpose of determining the effect of the various methods of irrigation on the growth of the different crops.

Crop	Method of irrigation	Height of plants (inches)	Weight of roots (pounds)	Weight of tops (pounds)	Weight of seed (pounds)
Wheat	Irrigated	1.250	0.150	0.450	0.050
	Not irrigated	1.100	0.120	0.400	0.040
	Irrigated	1.300	0.180	0.480	0.060
	Not irrigated	1.150	0.130	0.420	0.045
	Irrigated	1.200	0.160	0.460	0.055
Barley	Irrigated	1.100	0.100	0.350	0.030
	Not irrigated	1.000	0.080	0.300	0.020
	Irrigated	1.150	0.110	0.380	0.035
	Not irrigated	1.050	0.090	0.320	0.025
	Irrigated	1.120	0.105	0.360	0.032
Oats	Irrigated	1.000	0.080	0.300	0.020
	Not irrigated	0.900	0.060	0.250	0.015
	Irrigated	1.050	0.090	0.320	0.025
	Not irrigated	0.950	0.070	0.270	0.018
	Irrigated	1.020	0.085	0.290	0.022



The results of the experiments conducted at the Central Laboratory of the Ministry of Agriculture, in 1901, for the purpose of determining the effect of the various methods of irrigation on the growth of the different crops, are shown in the following table. The table shows the height of the plants, the weight of the roots, the weight of the tops, and the weight of the seed, for each crop and method of irrigation. The results show that irrigation has a beneficial effect on the growth of all the crops, and that the effect is more pronounced in the case of wheat and barley than in the case of oats.

E S S A Y,
CONTAINING AN
A C C O U N T
OF THE
PROGRESS from the REVOLUTION,
AND THE
PRESENT STATE,
OF
P O P U L A T I O N
IN
ENGLAND AND WALES.

By RICHARD PRICE, D. D. and F. R. S.

MISS A. Y.

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ACCOUNT

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PROGRESS OF



PRESENT STATE

OF THE

ENGLAND AND WALES

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OF THE UNIVERSITY OF OXFORD

IN TWO VOLUMES

VOLUME I

LONDON: PRINTED BY J. JOHNSON, ST. PAUL'S CHURCH-YARD, 1793.

OBSERVATIONS

ON

The Progress (since the Revolution)
and the present State of Population
in ENGLAND and WALES.

IT will be proper to introduce these observations with the following accounts of LONDON and MIDDLESEX.

Number of Houses in LONDON, SOUTHWARK, WESTMINSTER, and the COUNTY OF MIDDLESEX, in the Year 1777; from the Accounts of the Surveyors of the House and Window Duties.

Houses charged in 1777, having 25 windows and upwards	—	—	12,560
Houses charged, having less than 25 windows	—	—	61,080
Total of houses charged	—	—	73,640
Uninhabited houses chargeable	—	—	3,368
Total of houses charged and chargeable	—	—	77,008
Cottages not charged by reason of poverty	—	—	13,562
Total of houses	—	—	90,570

Number of Houses in London, Southwark, Westminster, and the County of Middlesex, from the Survey mentioned by Dr. Brackenridge in a Paper read to the Royal Society in March 1758, and published in the Philosophical Transactions, vol. 50, p. 471.

Houses charged to the house and window tax in 1757	—	—	63,480
Houses uninhabited	—	—	4,810
Total of houses charged and chargeable			— — 68,290
Cottages	—	—	19,324
Total of houses, including cottages			— — — 87,614

R E M A R K S.

These accounts shew, that the number of houses in *London, Westminster, Southwark, and all Middlesex* had, in the course of about 20 years preceding 1778, increased 2,956 in the whole; but that the houses excused on account of poverty had decreased 5,762; from whence it follows, that the houses *charged* and *chargeable* had increased 8,718. —It should be considered, that most probably this is less than the real increase of the

the best sort of houses ; for the decrease of the cottages proves, that the meanest of the houses * which pay the tax must likewise have decreased ; and this decrease is to be added to 8,718, in order to obtain the whole increase of the best houses ; for it is obvious that, if the best houses had not increased as much as the worst decreased, the total of houses, instead of being greater in 1777, must have been less.—Perhaps, therefore, we shall reckon moderately enough if we reckon an increase within the last 20 years of 10,000 substantial houses in and about *London* ; and this is a number that falls little short of the whole number of houses in *Liverpool* and *Manchester*.

The increase of buildings in *London* has for several years been the object of general

* That is, houses paying the house duty of 3s. only. The number of these houses in 1777 was 5,738 ; but I have no account of it for any preceding year. It will appear presently, that taking *England* in the gross, there has been a great decrease in these houses ; and this makes it almost certain they must have decreased in *Middlesex*.—The decrease of cottages, or houses excused, since 1757, is the more remarkable, because the house and window duties have been increased since that year by three different acts of parliament, the first in 1758, the second in 1762, and the third in 1766.

observation. It deserves particular notice that it is derived entirely from the increase of luxury; an evil which, while it flatters, never fails to destroy. It has been shewn from authentic accounts, that the decrease of the lower people in *London* and *Middlesex* has kept pace with the increase of buildings. The annual deaths also in the Bills of Mortality have for many years been decreasing, and are now near 6,000 *per annum* less than they were fifty years ago. In particular; it is observable with respect to that part of *London* which lies within the city walls, that, though always filled with houses, the births and burials, and, consequently, the inhabitants *, have decreased ONE HALF.—The just account of this must be, that those who cannot now satisfy themselves without whole houses, or, perhaps, two or three houses, to live in, used formerly to be satisfied with lodgings, or with parts of houses.

The number of houses in *London*, *Westminster*, and all *Middlesex*, in 1690, was

* See a particular account of this fact in my *Observations on Reverfionary Payments*, page 190, 3d edit.

111,215, according to Dr. Davenant's account from the Hearth-books *.

I will only further observe concerning the preceding accounts, that they demonstrate that the number of inhabitants in London has been greatly over-rated. They have been sometimes estimated at a million. In an Essay on the State of London, on Population, &c. in the Treatise on Reversionary Payments, I offered evidence, which I thought little short of demonstration, to prove that they fell short of 651,000. But it now appears that, allowing 6 to a house †, and including the whole county of Middlesex, their number in 1777 was only 543,420.

That six to a house for London, and five to a house for all England, is too large an allowance, will be proved by the following recital of facts.

* See Dr. Davenant's works, vol. 1st, page 38. This number does not include *Southwark*.—The average of burials for five years in *London* before the present year, or 1779, was 20,835. The average for five years before 1690 was 22,742; that is, considerably greater than it has been for the last five years, though twelve parishes, now the most populous, were not then included in the Bills.

In NORWICH, according to a survey in 1752	Houses,	Inhabitants,	To a house,	5.
Shrewsbury, by a survey in 1750,	—	7,139	36,169	—
Northampton, by a survey in 1746	—	3,078	13,328	4 $\frac{1}{2}$.
The parish of Ackworth, Yorkshire, in 1767	—	1,083	5,136	4 $\frac{1}{2}$.
Newbury, Berkshire, in 1768	—	184	728	4.
Speen, adjoining to Newbury, in 1768	—	930	3,732	4.
Aldwinckle, Northamptonshire, in 1772	—	303	1,200	4.
The parish of Holy Cross, near Shrewsbury, in 1760	—	96	402	4 $\frac{1}{2}$.
Altringham, Cheshire, in 1772	—	242	1,050	4 $\frac{1}{2}$.
The Parish of St. Michael's, Chester, in 1772	—	248	1,029	4 $\frac{1}{2}$.
The town and parish of Bala, North-Wales, in 1774	—	127	618	4 $\frac{1}{2}$.
Fifty-nine Dutch villages mentioned by Struyk	—	401	1,723	4 $\frac{1}{2}$.
Birmingham, in 1770	—	12,005	45,888	3 $\frac{1}{2}$.
Liverpool, in 1773, including 400 in the Poor-house	—	6,025	30,804	5 $\frac{1}{2}$.
Manchester and Salford, in 1773	—	6,340	34,407	5 $\frac{1}{2}$.
Leeds, in 1775	—	4,338	27,246	6 $\frac{1}{2}$.
	Families	4,096	17,121	To a family 4 $\frac{1}{2}$.

In the District of Vaud in Switzerland -						
Chester, in 1774	—	—	—	—	112,951	To a family, $4\frac{1}{2}$.
Rome, in 1770	—	—	—	—	14,713	$4\frac{1}{6}$.
Calne, Wiltshire	—	—	—	—	158,442	$4\frac{1}{4}$.
Liverpool	—	—	—	—	3,467	$4\frac{1}{2}$.
Manchester	—	—	—	—	34,407	$4\frac{1}{6}$.
Bolton in Lancashire, in 1773, including Little Bolton	—	—	—	—	27,246	$4\frac{1}{2}$.
Bury in Lancashire, in 1772	—	—	—	—	5,339	To a house, $4\frac{1}{2}$.
The parish of Bala in North-Wales, in 1774	—	—	—	—	2,090	$4\frac{1}{2}$.
Chippenham, Wilts, in 1773	—	—	—	—	1,723	$4\frac{1}{6}$.
Brenhill, near Calne, in Wiltshire	—	—	—	—	2,407	5.
The Island of Sicily (see end of 2d vol. of Brydone's Travels)	—	—	—	—	1,206	$5\frac{1}{2}$.
Fourteen market towns mentioned by Dr. Short, Comparative History, page 58	—	—	—	—	1,123,163	$4\frac{1}{2}$.
Sixty-five country parishes, <i>ibid.</i>	—	—	—	—	97,611	To a family $4\frac{1}{2}$.
	—	—	—	—	76,284	$4\frac{1}{2}$.
	—	—	—	—		In

In the Parish of <i>Stelton</i> , Yorkshire, in 1777	— Houses	109	— Inhabitants,	506	— To a house, $4\frac{1}{2}$.
The town and parish of Wycombe, Bucks	— Families	500	—	2,461	— To a family, $4\frac{2}{5}$.
Worley, Barton, Pendleton, Pendlebury, and Clifton, Lancashire, in 1778	— Families	1,685	—	9,117	— $5\frac{1}{2}$.
Parish of St. Cuthbert, Edinburgh, in 1743 (see Maitland's History of Edinburgh, page 171)	— Families	2,370	—	9,731	— $4\frac{1}{10}$.
In a number of small towns and parishes in the Generalities of Auvergne, Lyon, and Rouen, in France (see <i>Recherches sur la Population</i> , par M. Meffiance, pages 8, 26, and 62)	— Families	24,910	—	99,002	— 4.
Parish of Manchester, exclusive of the town, in 1774	— Families	2,525	—	13,786	— $5\frac{1}{2}$.
Parish in the city of London (see Phil. Trans. part 2d, page 796)	— Houses	2,412	—	13,786	— To a house, $5\frac{1}{10}$.
	—	—	—	—	— 6 to a house.

*Number of Houses in England and Wales,
from the Returns of the Surveyors of the
House and Window Duties in 1766 and
1777.*

	In 1765.—	In 1777.
Houses charged, having 25 windows and up- wards — — —	*32,595—	32,595
Houses having 21, 22, 23 and 24 windows —	12,404—	14,623
Total of houses having more than 20 windows	44,999—	47,218
Houses having from 12 to 20 windows —	88,494—	98,756
Total of houses having more than 11 windows	133,493—	145,974
Houses having 8, 9, 10, and 11 windows —	102,525—	117,857
Total of houses having more than 7 windows	236,018—	263,831
Increase in 1777 of houses having from 8 to 24 windows — — —		27,813

* In the returns for 1765 this number is wanting. I have, therefore, supposed it the same that it was found to be in 1777. But the truth is, that it must have been less, as will appear presently.

Houses

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	In 1765.	In 1777.
Houses charged having 7 windows — —	*	— 131,950
Total of houses paying the window tax —	236,018	— 395,781
Houses paying only the house tax of 3s. —	442,897	— 286,296
Total of houses charged	678,915	— 682,077
Increase in 1777 of houses charged — —	3,162	
Houses uninhabited, but chargeable —	† 25,628	— 19,396
Total of houses charged and chargeable —	704,543	— 701,473
Decrease in 1777 of houses charged and chargeable — —	3,070	
Cottages excused on account of poverty only	276,149	— 251,261
Total of houses charged, chargeable, and excused — — —	980,692	— 952,734

* The number of houses in 1765, having exactly seven windows, was 400,273; but by the law, as it then stood, all such houses were exempted from the window tax. In 1766 the tax was extended to these houses; and the consequence was, that near two thirds of them were reduced to houses having only six windows.

† The decrease which may be here observed in the number of empty, but chargeable, houses, is an effect which could not but attend the greater demand for houses which produced the increase between 1765 and 1777, of houses having more than seven windows.

Decrease

Decrease of houses charged, charge-
able, and excused, from 1765 to

1777 — — — — — 27,958

To this decrease add the increase of
houses having from 8 to 24 win-

dows, or — — — — — 27,813

And the total will shew, that the number
of houses not having *eight* windows was
55,771 less, in 1777, than it had been in
1765.

Again; from 27,813, the increase in
1777 of houses having from 8 to 24 win-
dows, subtract 3,162, the increase of houses
charged, having less than 25 windows; and
it will appear, that in the houses charged,
having 7 windows or less, there has been in
the same period a decrease of 24,651 houses.

—But this is by no means the whole de-
crease of houses of this sort. The increase
of houses having more than 24 windows
ought to be added; but the number of such
houses not having been given in the return
for 1766, it does not appear what this in-
crease has been. It seems, however, past
doubt, that there must have been such an
increase, because all other houses having
more than seven windows had increased.

NUMBER OF HOUSES IN ENGLAND AND
WALES in 1759, from the Return of the
Surveyors of the House and Window Duties.

Houses charged in 1759	— — — —	679,149
Uninhabited houses in 1759 charge- able	— — — —	24,904
Houses excused on account of po- verty only	— — — —	282,429
Total of houses in 1759	— — — —	986,482
— — — — — in 1765, see p. 286,		980,692
— — — — — in 1777, see p. 286,		952,734
Diminished in seven years from 1759	— — — —	5,790
Diminished in 18 years from 1759	— — — —	33,748
Number of houses charged in 1750 *	— — — —	690,000
Number of houses charged in 1710 †	— — — —	729,048
		Total

* This rests on the authority of Dr. Brackenridge. His words are, (in Philosophical Transactions for 1756, page 887) "I have been certainly informed, that from the survey lately made of the window lights after 1750, there are about 690,000 houses charged to that tax in England and Wales, besides cottages that pay nothing."

† I give this number also from the authority of Dr. Brackenridge, who took it himself from the accounts of the

Total of houses according to the
Hearth-books of Lady-day,

1690 * — — — — 1.319,215

Total of houses from the

Hearth-books in 1666† — 1.230,000

the Tax Office. See a letter to George Lewis Scot in the Philosophical Transactions, vol. 49, part 1st, p. 270. —By an act of the 8th of *William*, all houses of 10 windows or more paid 6s. over and above 2s. for the house; and all houses having 20 windows or more paid 10s. In 1710, certain additional duties were laid on all houses having 10 windows and upwards; and with a view to these duties, a new assessment was ordered, and a return made of the houses subject to an assessment. — This return could not include in it cottages, nor probably did it include uninhabited houses.

* This account is given on the authority of Dr. Davenant. See his works, vol. 1st, page 38, where the number of houses, and also of hearths, is given separately for each county. — In page 136 he says, that “ the hearth-tax had given a view *certain enough* of the number of families in the kingdom.”

† See Tindall’s Continuation of Rapin’s History, vol. 1st, page 53. — Dr. Davenant says, that from 1666 to 1688 there had been about 70,000 new foundations laid. See his works, vol. 1st, page 370. — It is probable that the civil war in the time of King Charles the First, and the emigrations which then took place, lessened the number of people in the kingdom; and therefore, in in Queen *Elizabeth*’s time, or about the *Reformation*, the number of inhabitants in *England* might have been greater than it was even at the *Revolution*, agreeably to the facts mentioned at the end of my *Appeal to the Public on the Subject of the National Debt*, page 87, &c.

OBSERVATIONS *on the foregoing Accounts.*

First. The first of these accounts makes the number of houses in England and Wales in 1777 to be 952,734. Let it, however, be stated at a MILLION. Five persons to a house is too large an allowance, as appears from the accounts in page 281. It follows, therefore, that the number of inhabitants in *England and Wales* must be short of FIVE MILLIONS.

In the kingdom of SWEDEN the number of inhabitants was 2.446,394, in 1763.—In the kingdom of NAPLES (one of the *Two Sicilies*) it was 4.311,503, in 1777.—In all FRANCE, 25.741,320, in 1772*.

These

* The account here given of *Sweden* is taken from actual surveys of the kingdom in 1757, 1760, and 1763. In the first of these years the inhabitants, of all ages, were found to be 2.323,195; in the second, 2.367,498; in the third, 2.446,394. See a Memoir by M. Wargentin in the 15th vol. of the *Collection Academique*, printed at Paris, 1772. The account of the kingdom of *Naples* is also given from surveys made there every year, and published in the Court Calendars.—In 1766, the number of inhabitants was 3.771,234; in 1772, 4.040,680; in 1777, 4.311,503.

The Intendants of provinces in *France* were, in 1770, 1771, and 1772, ordered to make returns of the number of deaths, births, and marriages in their respective districts. The annual average of deaths for these three years was 780,040. See a *Treatise On the Legislation*
and

These facts shew, in a striking light, the superiority which arts, commerce, science, industry, and liberty give to a people.

ENGLAND

and Commerce of Corn, printed at Paris in 1775, and translated into English, and published in London in 1776, page 42.—I have been assured by the ingenious author, now the Director-general of the finances of *France*, that this account may be depended on as rather below the truth; and it affords a decisive proof that the number of inhabitants in *France* cannot be less than that stated above, or 25,741,320, which is the product of the average of deaths multiplied by 33. That this is the least multiplier which ought to be used will appear undeniably from the following facts.—In *Sweden*, the average of deaths for 9 years ending in 1763, was 69,125, or a 35th part and two-fifths nearly of the inhabitants. See M. Wargentin's Memoir just referred to.—In the kingdom of *Naples*, the average of deaths for 5 years before 1778, was 115,412, or a 37th and a third of the inhabitants.—These facts (and many others of the same kind may be found in the Treatise on Reversionary Payments, page 200) convince me that the average of annual deaths in *France* might have been multiplied by 35 instead of 33, and this would have brought out the number of inhabitants 27,301,400.—The same conclusion nearly may be drawn from the births in *France*, the average of which for five years ending in 1774, was 928,918. See *Recherches sur la Population de la France*, par M. Moheau, printed at Paris in 1778, page 147.—In *Sweden*, the average of annual births for 9 years, ending in 1763, was 90,240, or a 27th part and a tenth of the inhabitants.—In the kingdom of *Naples*, the average of annual births for 5 years, ending in 1777, was 166,808, or a 25th part and four-fifths of the inhabitants. The medium is $26\frac{1}{2}$, which

ENGLAND does not consist of many more inhabitants than the kingdom of NAPLES; but in respect of dignity, weight and force, the kingdom of NAPLES, compared with it, is *nothing*. Not long ago, this little island, with its dependencies, like the state of ATHENS formerly among the *Greeks*, was the arbiter of EUROPE, and more than a match for all the three kingdoms I have mentioned, with SPAIN added to them.

Secondly. The great disparity between the numbers of people in the higher and the lower ranks of life seems to deserve particular observation, as it may be collected from the foregoing accounts. Families living in houses having *seven* windows or less, must consist of persons in the lowest stations; and yet the number of these houses was 688,903 in 1777. Add to these such of the lowest people as live in the remaining 263,903 houses; and it will appear, that the people of property and opulence in the state, compared with the rest, are indeed a

multiplied by 928,918, gives 24-616,327.—But it is certain, that a greater multiplier than $26\frac{1}{2}$ ought to be used in this case, because the births exceed the deaths considerably less in *France* than in either *Sweden* or *Naples*.—Upon the whole, therefore, I reckon that it appears with sufficient evidence that the inhabitants of *France* may very moderately be stated at the number I have given.

very small body. And yet their number is *now* greater in this country than it ever was; and, very probably, it is much greater in this country than in any other*.—It is proper to add, that this observation shews us distinctly why no taxes in a state can be very productive which do not reach the lower as well as higher ranks of people.

But, thirdly, What requires most to be attended to is the certain evidence which the preceding accounts give of the progress of depopulation in this kingdom.—The number of houses in ENGLAND and WALES was at the REVOLUTION 1,319,215. The number of houses now is not a *million*. Our people, therefore, since that æra have decreased near a *quarter*.—This appears distinctly, as far as Dr. Davenant's account is to be depended on †. The following facts and

* In ENGLAND, the houses having more than *seven* windows are above a *fourth* of all the houses. In SCOTLAND, the number of houses having more than *five* windows, and paying the house and window duties, was, in 1777, only 16,206; and consequently could not be above a *fifteenth* of all the houses.—Agreeably to this poverty, the people of SCOTLAND, though more than a *fifth* of *Britain*, do not contribute more than a *fiftieth* to the revenue.

† Some may suspect that Dr. Davenant has, by mistake, taken from the Hearth-books the number of
U 3
houses

and observations will confirm this account, and furnish us with some additional evidence on this subject.

First. It appears, that there has been a very great decrease, since the Revolution, in the produce of a tax called the *hereditary and temporary excise*. This excise (almost the only one that existed before the Revolution) consists chiefly of 2*s.* 6*d.* per barrel on all strong beer or ale above 6*s.* the barrel, and 6*d.* on every barrel of ale sold at 6*s.* or less; and also a duty of 2*s.* 6*d.* per hogshead on cyder and perry; a duty on mead, strong waters, and low wines and spirits. The gross annual produce of this tax for three years, ending at 1689, was (as appears from the Excise books) 740,147*l.*—Its gross annual produce for four years, ended in 1768, was 527,991*l.* It has decreased, therefore, 212,156*l.* *per annum*. One of the reasons of this decrease has been, that in 1736 the duties on low wines and spirits (amounting then to 70,000*l.* *per ann.*) were taken from the Hereditary and Temporary Excise, and carried to the Ag-

houses in the kingdom, when he ought to have taken the number of *families*. But this is improbable; and if true, will make no great difference, as may be inferred from the accounts in page 282.

gregate

gregate fund. Deduct*, therefore, 70,000*l.* from 212,156; and the real decrease will be 142,156*l.* And this decrease will appear more remarkable, when it is considered how much less the currency and wealth of the kingdom were before the Revolution than they are now.—It may be said, that more wine is now drank; but this, being confined to the higher class of people, makes no great difference.—It may with more reason be objected, that the lower people drink now greater quantities of spirituous liquors, and therefore less ale. With respect to this, it seems sufficient to observe, that it appears from the Excise books that the use of spirituous liquors never sunk the produce of this excise more than about 40,000*l.* in a year; and that since 1751 it has been so much checked by new regulations, additional duties, and other causes, that most probably it does not prevail much more now than it did at the Revolution.

* This is too great a deduction; for the use of spirituous liquors was in 1736 so much increased, that it became necessary to restrain it by additional duties.—The produce of that part of this Hereditary and Temporary excise which consists of the tax upon beer only, was 674,387*l.* in 1688; and 694,476*l.* in 1689. See Dr. Davenant's works, vol. 1st. page 175.

After allowing, therefore, for the operation of this cause *, (and also for the increased use of wine) there will remain a diminution unaccounted for, of at least 100,000 *l.* *per annum*.

In conformity to this fact, it appears that there has been a proportionable diminution in the quantity of beer brewed for sale, and in the number of victuallers.—For three years, ended in 1689, the annual average of strong barrels brewed for sale was 5.055,870. The average of small barrels was 2.582,248.—For three years, ended in 1768, the former † average was 3.925,131; the latter 1.886,760.

* The following facts will confirm what is here said, and shew the progress of gin-drinking in the kingdom.—The use of spirituous liquors prevailed most in 1750 and 1751; and the *annual* average of spirits drawn from malted corn, cyder, melasses, and brewers' wash in those two years was 11.326,976 gallons.—In 1767 and 1768 it had sunk to 3.663,568 gallons.—In 1730 and 1731, it was 6.658,788 gallons.—In 1692 and 1693, it was 2,329,487 gallons.

In 1767 and 1768 the annual average of excisable brandy imported was 1,612,631 gallons. In 1688 and 1689, it was 1,713,974.

† It is natural to suspect that this decreased consumption of beer must have been owing to the increase of the taxes upon it. But this does not appear; for in 1761, (after an addition in 1760 of 3*d.* per bushel to the duty on malt) an addition was made to this tax of 3*s.* *per*

1.886,760.—The average of common victuallers in the whole kingdom for the former three years * was 47,343; for the latter three years, 34,867.—This last fact seems of particular consequence, because victuallers in both periods include all that keep houses for selling any strong liquors; and because also there is reason to believe, that the private brewery †, of which no account

per barrel, and yet it produced in the following years rather more in proportion than it did before.—The quantity likewise of strong beer brewed for sale increased a little afterwards; though these two additions were so considerable as to bring into the revenue near 900,000*l.* per annum.

* For 10 years before the check given to the use of spirituous liquors in 1751, the victuallers in the kingdom amounted to near 48,000, though the quantity of strong beer brewed annually for sale was then less than it has been for the last 15 years. This, I suppose, must have been owing to the vast numbers of shops for selling gin, which, during that period, were opened every where.

† The number of common brewers in the whole kingdom in 1687 and 1689 was 776; in 1767 and 1768 it was increased to 1083. One reason of this must be, that fewer victuallers and private people now brew their own beer.—It is remarkable, that the number of brewers in London *decreased* during the same period from 187 to 157; and also that the quantity of small and strong beer brewed for sale *decreased* from 1.958,859 to 1.533,242 gallons. And this seems to confirm what has been already suggested, that even London is less populous now than it was at the Revolution. See page 281.

This

account is taken, was greater formerly than it is now.—I cannot help adding, as a farther fact, indicating a particular degree of populousness at the Revolution, that King William wanting, in 1689, to raise 23 new regiments for the war in Ireland, the levies were completed in six weeks. See Sir John Dalrymple's *Memoirs of Great Britain*, vol. 1st, page 384.—But what is most of all decisive in the present question is, the depopulation which has certainly taken place *lately* in this kingdom.

From the preceding accounts it appears, that between the years 1765 and 1777 a destruction has taken place of at least 55,771 houses having less than 8 windows; which is equal to the loss of above a *quarter of a million* of those inhabitants who furnish recruits for our navy and army, and trading ships; and who, therefore, constitute the main strength of the kingdom.

I am not sensible that any thing can be objected to the evidence from which this

This decrease was gradual and slow till 1726. After 1726 it became considerable; and for some years before 1750, the quantity of beer consumed in London was about 100,000 gallons *per annum* less than it is now, in consequence, undoubtedly, of the excessive use of spirituous liquors which then took place in *London* more than any where else,

conclusion

conclusion has been drawn, except that there is an uncertainty in the returns of the cottages. But it should be observed,

First, That this uncertainty does not at all affect the evidence for the diminution of houses *charged* having less than eight windows, and of which exact accounts are kept.

Secondly, The returns of the cottages have not, I suppose, been made with less care for 1777 than for 1765; and it is the difference only on which the conclusion I have drawn depends.

But, thirdly, The diminution which there has certainly been in the houses *charged* having less than eight windows, proves undeniably, that there must have been a proportionable decrease in the cottages *not* charged.

Between the years 1759 and 1765 there appears in the returns, as I have been able to give them, only a diminution of 234 in the houses charged, and of 6,280 in the cottages. See pages 286, 288. It should not, however, be forgotten, that the increase of houses having *more* than seven windows, which took place between 1765 and 1777, must have begun before 1765; and that this

increase ought to be added to the whole decrease between 1759 and 1765, in order to determine the whole decrease in that period of the meanest sort of houses; and such an addition would shew distinctly the rate and progress of depopulation among us between the years 1759 and 1777.

Before 1759, no more appears than that the houses *charged* had diminished 10,851 between 1750 and 1759; and 49,899 between 1710 and 1759. If to this decrease is added the decrease that must have been going on faster all this time in the houses *not* charged, it will appear too probable that the depopulation since the REVOLUTION cannot have been less than I have stated it in page 293. It would have been considerably greater, had it proceeded from that period constantly at the rate with which we know it has proceeded since 1765. But there is reason to believe that of late it has been much accelerated by particular causes, some of which will be mentioned presently.

The Honourable Mr. Grenville, in a pamphlet entitled *Considerations on the Trade and Finances of the Kingdom*, after giving the same account with that here given of the houses in *England and Wales* in 1759 and 1765, expresses the utmost surprize at the

the proofs of depopulation which it afforded, and observes, "that the destruction of 5790 houses in so short a space as eight years, is such a symptom of distress as requires every attention to check the progress of the evil.—Relief to the landed interest is now (he adds) no longer the concern of individuals only who are to receive that relief, but is become an important national concern."—What would he have said, had he known that the depopulation which shocked him was proceeding so rapidly as I have shewn; that no attention would be given to it; that the public burdens, instead of being lessened, would increase; and that he himself had laid the foundation of such an increase of them as would, in a few years, bring the nation to the brink of ruin?

The increase in the higher classes of houses has been for some time obvious to every one. It may be imagined, that this implies such an increase of people in the middle and higher ranks of life, as makes amends for the depopulation among the lowest ranks. But the truth is, that no such conclusion can be drawn. One of the principal causes of this increase has been that very evil which has destroyed the common people,

people, or the increase of luxury. This, I think, has been demonstrated, by the account I have given of London *. See page

* At Birmingham, in 1700, the inhabitants were 15,042, the houses 2,504; or 6 to a house.—In 1750, the inhabitants were 23,688; the houses, 4170, or $5\frac{7}{8}$ to a house.—In 1770, the inhabitants were 30,804; the houses, 6,025, or $5\frac{1}{9}$ to a house.—In this account we see the gradual progress of luxury at *Birmingham*; the houses having increased so much faster than the inhabitants, that 600 houses in 1770 contained no more people than 511 contained in 1700.—See *Treatise on Reversionary Payments*, page 427.

The following circumstance may perhaps deserve some notice here.—By the new regulations of the window-tax in 1776 particular inducements were given to divide buildings deemed *single* houses, but holding *several* families, into houses having only one family in each; and this, as well as luxury, may have contributed to increase the number of houses without increasing the number of inhabitants.

For instance. By dividing a house having 30 windows, and containing *three* families, into *three* houses or tenements, having ten windows, and one family in each house, only 9*s.* *per annum* would have been saved before 1766; but since the alteration in the tax that year, 1*l.* 14*s.* *per annum* may be got by such a division.—In like manner. By dividing such a house into two houses, having one family in each, and 15 windows, 3*s.* *per annum* would have been *lost* before 1766; but now 15*s.* *per annum* may be *saved* by it.

N. B.—Before 1766, houses having from eight to eleven windows paid 1*s.* *per window*; and houses having more than eleven windows paid 1*s.* 6*d.* *per window*, besides

page 280. It must, however, be acknowledged, that in many of our towns, and particularly our manufacturing towns, there has been a great increase of people as well as of houses; but it should be considered, that it has been derived from the depopulation of country parishes and villages, the inhabitants of which, by removing to these towns, and many of them thriving there, and living in better houses, have increased the number of such houses at the expence of meaner houses. This increase of people, therefore, in our towns has either quickened depopulation; or, if not, it must have been owing entirely to the increase of trade. From the accounts of the exports at the Custom-house it appears, that* for some years before 1765 they were at the highest, and that they

besides 3s. for the house.—By the new regulations in 1766, besides the old duty of 3s. for every house, all houses having seven windows pay 2d. per window. Houses having 8, 9, &c. to 13 windows, pay respectively 6d.—8d.—10d. &c. to 1s. 4d. per window.—Houses having from 14 to 19 windows pay 1s. 6d. per window.—Houses having 20, 21, &c. to 24 windows, pay 1s. 7d.—1s. 8d. &c. to 1s. 11d.—Houses having above 24 windows, pay 2s. per window.

* See *The Additional Observations on Civil Liberty*, page 113. The annual average of exports for four years ending in 1764, was 15,793,158*l.*—In 1773, the average for nine years had sunk to 14,814,074*l.* But the imports had

they have since decreased. This decrease, however, has been more than compensated by the encrease of our *home-consumption*, occasioned by a vast encrease* of luxury; and this, though it has operated fatally among the body of the lower people, has, in one way contributed to retard the progress of depopulation; I mean, by furnishing an increase of employment, and consequently of the means of subsistence, for our manufacturers and artizans. But though depopulation has been thus checked, yet it has proceeded rapidly; and if we ascribe one half of the increase in the higher classes of houses to this cause (or a real increase of people)

had increased from 10.£10,870*l.* to 11.996769*l.*—The decay of foreign trade may farther be understood from hence. In 1764, the drawbacks on exportation amounted to 2.264,820*l.*—The average for ten years after 1764 was 1.843,404*l.*—but in 1776 they sunk to 1.544,300*l.*—In 1777, to 932,860*l.*—In 1778, to 868,600*l.*

* The following account will shew how great this increase has been.—The *net annual* amount of all the excise duties for two years, ending 1768, was 4.431,075*l.* For two years, ending in 1773, it was 4.712,265*l.*—For two years, ending in 1777, it was above FIVE MILLIONS, after deducting the new taxes for 1776 and 1777.—The great increase of our importations, while the exportations have decreased, as mentioned in the last note, is another certain proof of the increase of luxury; and has probably been the means of turning the balance of trade against us. See *Additional Observations on Civil Liberty*, p. 116, &c.
and

and the other half to luxury, as before explained, we shall, I think, reckon very moderately; and it will appear, that in eleven years near 200,000 of our common people have been lost.

I will only observe farther, that since the Revolution, most of the causes of depopulation have prevailed so much as to render it an evil which could not but happen. The causes I mean are—the increase of our navy and army, and the constant supply of men necessary to keep them up—a devouring capital, too large for the body that supports it*—the three long and destructive continental wars in which we have been involved—the migrations to our settlements abroad, and particularly to the East and West Indies—the engrossing of farms—the high price of provisions—but above all, the increase of luxury, and of our public taxes and debts.

I have given a particular account of these causes of depopulation in the Supplement to the Observations on Reversionary Payments,

* PARIS cannot contain so much as a *fiftieth* part of the inhabitants of *France*. LONDON contains a *ninth* of the inhabitants of *England*; and consumes *annually* about 7,000 persons, who remove into it from the country every year, but without increasing it.

page 371, third edition.—I will here only observe, that the depopulation they have produced is the more mortifying, because it seems, in some degree, peculiar to this nation.—In FRANCE, (in the principality of *Dombes*, the diocese of *Vaison*, and the seven generalities of *Auvergne**, *Lyon*, *Rouen*, *Bourgogne*, *Provence*, and *Alençon*, containing 2152 parishes) the average of annual births before 1764 had increased in 60 years from 54,827 to 59,894, or in the proportion of 100 to 109.—The average for five years of annual births in the whole kingdom of *France*, (as mentioned in the note, page 281) had been 928,918, in 1774, of which 479,649 were males, and 449,269 females.—The average of deaths, as mentioned in the same note, had been 780,040 for *three* years, ending in 1772. But Mr. MOHEAU has given the average for *five* years, ending in 1774†; and it was 793,931. The annual

* See *Recherches sur la Population*, printed at Paris in 1766, page 274, and page 19, &c. See also on this subject M. MOHEAU's *Recherches & Considerations sur la Population de la France*, printed at Paris in 1778; where, in page 276, &c. the account of the increase of the generalities of *Auvergne*, *Lyon*, and *Rouen* is continued to 1774.

† MOHEAU's *Recherches*, &c. page 65.—The average of marriages was 192,180.

excess of the births above the deaths was, therefore, 134,987; or near a *seventh* of the births; and this is probably an excess which in *France* more than counterbalances the destruction occasioned by emigration, war, and the sea-service.

The increase in SWEDEN and the kingdom of NAPLES has been distinctly mentioned in the note just referred to.

In the English colonies in NORTH AMERICA there has for many years been an increase scarcely ever before known among mankind.

Thus unhappily distinguished are we in this country. Nor will it appear wonderful, when we consider how unhappily we are distinguished by some of the worst causes of depopulation; and with what particular force they have been operating for the last *twenty* years. At present we are sinking under new incumbrances and difficulties. The most valuable of our dependencies are lost. Another foreign war is begun. Trade is declining; our strength is wasting; and at the same time, that load of debts which has pressed so heavily on our population, is increasing faster than ever.—Never, certainly, were the resources of a state so anti-

cipated and mortgaged*.—Never before did imprudence and extravagance bring a great kingdom into such peril.

* The terms of the loan for the present year will throw some light on what is here said.—A 3 per cent. stock has been sold at 40 per cent. discount, to which has been annexed an Annuity of $3\frac{1}{4}$ per cent. for 29 years, reckoned at $11\frac{1}{4}$ years purchase, but really worth (when the 3 per cents. are at 40 per cent. discount) $15\frac{1}{10}$ years purchase.—The public, therefore, besides subjecting itself to the necessity of paying at redemption 40 l. more than it has received for every 100 l. stock, has given a present premium on the short Annuity of near 27 per cent. And even on these terms, (with the profits of a lottery added) only seven millions could be got, though near nine millions and a half were wanted for defraying the necessary expences of the year, exclusive of the usual vote of credit for a million.—These deficiencies must be made good; and at least ten or eleven millions more borrowed within the next eight or nine months, for which, very probably, if the war continues and spreads, a higher interest and still higher premiums must be given.—The national debt is now considerably greater than it was in 1776, when Mr. Hume wrote the words quoted in the next page; and it is advancing fast towards two hundred millions. It may signify little how a nation, in such circumstances, borrows money; but I am mistaken if I have not (in the *Supplement* to the *Additional Observations on Civil Liberty*) proposed regulations by which the loan of this year might have been procured at an interest of 5 (or, at most, $5\frac{1}{2}$) per cent. and consequently an expence of 100,000 l. per annum for 29 years saved; which saving, properly applied, might have discharged, in 28 years, either the capital of five millions bearing four per cent. interest created in 1777, or a larger capital in the three per cents.

“ Our

“ Our late delusions (says Mr. HUME *)
“ have much exceeded any thing known in
“ history, not even excepting those of the
“ Crusades. For there is no arithmetical
“ demonstration that the road to the Holy
“ Land is not the road to Paradise ; as there
“ is, that the endless increase of national
“ debt is the direct road to national ruin.
“ ———So egregious, indeed, has been our
“ folly, that we have even lost all title to
“ compassion under the numberless calamities
“ that are waiting us.”

* See History of England, vol. 5th, page 475.

May 20, 1779.

P O S T.

P O S T S C R I P T.

The Favour of a Friend has lately procured for me, from the Tax-Office, the following Particulars in the Returns for 1759 and 1756, mentioned in Page 288.

	In 1756.	In 1759.
Houses <i>charged</i> having less than 10 windows	482,533	— 475,147
Houses <i>charged</i> having from 10 to 14 win- dows — — —	105,153	— 103,610
Houses <i>charged</i> having from 15 to 19 win- dows — — —	55,457	— 53,193
Houses <i>charged</i> having 20 windows or <i>more</i>	47,559	— 47,199
Total — — —	690,702	— 679,149

This account scarcely needs a comment. A comparison of it with the *returns* in page 285, for 1765 and 1777, will shew distinctly, that *before* 1759, houses of all sorts were decreasing; but that *before* 1765, an increase (produced by increased trade and luxury, as explained page 301, &c.) begun among the *higher* classes of houses; which,

which, *after* 1765, became very considerable; but was all along accompanied with a decrease much more considerable in those inferior classes of houses which constitute near *four fifths* of all the houses in the kingdom.

N. B. In page 288, the year 1750 should have been 1756. The number also 690,000 should have been 690,702. And in conformity to these corrections, the words in page 300, *diminished* 10,851 *between* 1750 *and* 1759, should be read, *diminished* 11,553 *between* 1756 and 1759.



THE END.

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